Phase Estimation Algorithm

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Rotation Matrix

We use a rotation matrix

$$U = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

with $c = \cos(\alpha)$, $s = \sin(\alpha)$ and a real-valued angle α as an example. U has eigenvalues

$$\lambda_\pm = c \pm \mathrm{i}s = e^{\pm i\alpha}.$$

Thus, $\phi = \alpha/(2\pi)$. The corresponding eigenvectors are of the form

$$u_\pm = \begin{pmatrix} 1 \\ \pm\mathrm{i} \end{pmatrix}.$$

Phase Estimation

We use

```
t=6
```

in the second register which allows us with probability $1-\epsilon$ to get the correct phase up to $t - \left\lceil \log\left(2 + \frac{1}{2\epsilon}\right)\right\rceil$ digits. Let us choose

```
epsilon <- 1/4
## note the log in base-2
digits <- t-ceiling(log(2+1/(2*epsilon))/log(2))
digits
```

```
[1] 4
```

and therefore expect an error of less than

```
2^(-digits)
```

```
[1] 0.0625
```

We start with qubit 1 in state u_+

```
x <- S(1) * (H(1) * qstate(t+1, basis=""))
```

and we define the gate corresponding to U

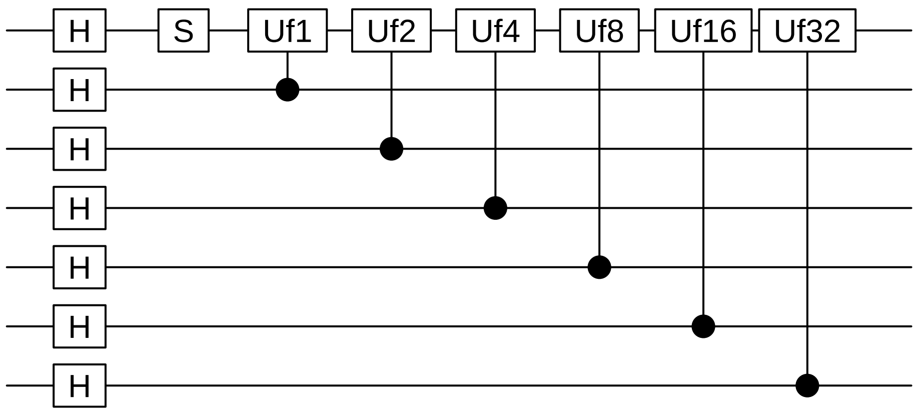
```
alpha <- pi*3/7
s <- sin(alpha)
c <- cos(alpha)
## note that R fills the matrix columns first
M <- array(as.complex(c(c, -s, s, c)), dim=c(2,2))
Uf <- sqgate(bit=1, M=M, type=paste0("Uf"))
```

Now we apply the Hadamard gate to qubits 2,. . . ,t+1

```
for(i in c(2:(t+1))) {
  x <- H(i) * x
}
```

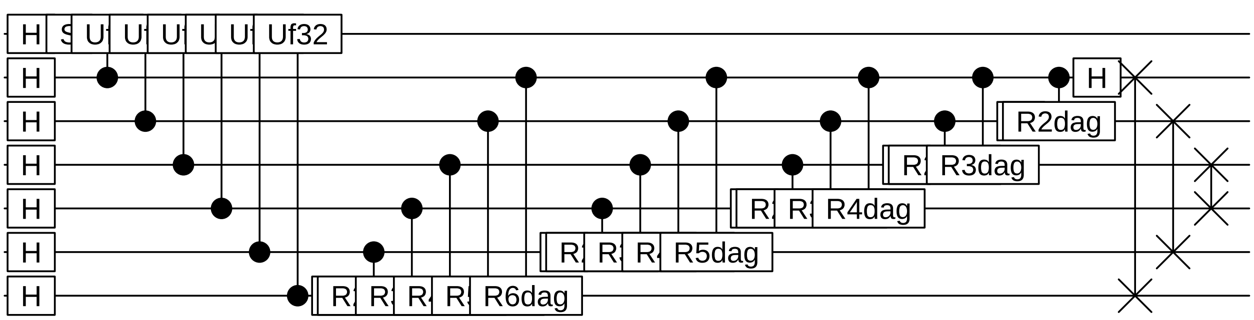
and the controlled U_f

```
for(i in c(2:(t+1))) {
  x <- cqgate(bits=c(i, 1),
              gate=sqgate(bit=1,
                          M=M, type=paste0("Uf", 2^(i-2)))) * x
  M <- M %*% M
}
plot(x)
```



Next we apply the inverse Fourier transform

```
x <- qft(x, inverse=TRUE, bits=c(2:(t+1)))
plot(x)
```



x is now the state $|\tilde{\varphi}\rangle|u\rangle$. $|\tilde{\varphi}\rangle$ is not necessarily a pure state. The next step is a projective measurement of $|\tilde{\varphi}\rangle$

```
xtmp <- measure(x)
cbits <- genStateNumber(which(xtmp$value==1)-1, t+1)
phi <- sum(cbits[1:t]/2^(1:t))

cbits[1:t]
```

```
[1] 0 0 1 1 0 1
```

```
phi
```

```
[1] 0.203125
```

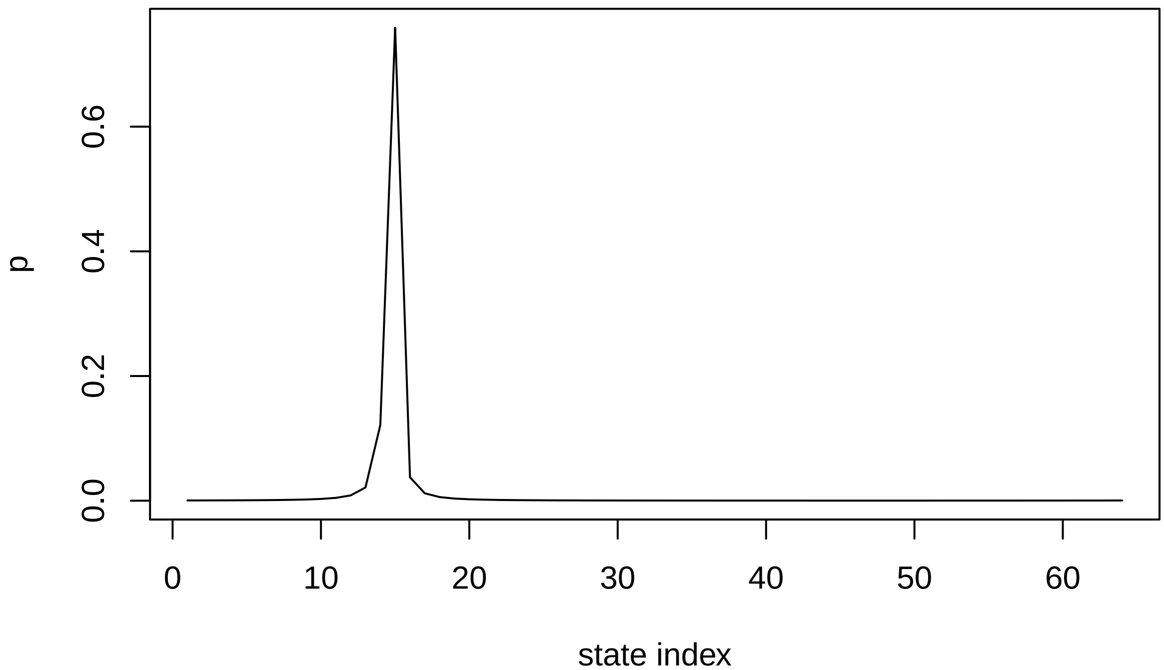
Note that we can measure the complete state, because $|u\rangle$ is not entangled to the rest. We find that usually

```
phi-alpha/(2*pi)
```

```
[1] -0.01116071
```

is indeed smaller than the maximal deviation $2^{-\text{digits}} = 0.0625$ we expect. The distribution of probabilities over the states in $|\tilde{\varphi}\rangle$ is given as follows (factor 2 from dropping $|u\rangle$)

```
plot(2*abs(x@coefs[seq(1,128,2)])^2, type="l",
     ylab="p", xlab="state index")
```



Starting from a random state

The algorithm also works in case the specific eigenvector cannot be prepared. Starting with a random initial state $|\psi\rangle = \sum_u c_u|u\rangle$, we may apply the very same algorithm and we will find the approximation to the phase φ_u with probability $|c_u|^2(1-\epsilon)$.

We prepare the second register in the state

$$\begin{pmatrix}1\\1\end{pmatrix} \;=\; (1-i)u_+ + (1+i)u_- \,.$$

```
x <- (H(1) * qstate(t+1, basis=""))
```

This implies that we will find both φ_u with equal probability.

```
for(i in c(2:(t+1))) {
  x <- H(i) * x
}
M <- array(as.complex(c(c, -s, s, c)), dim=c(2,2))
for(i in c(2:(t+1))) {
  x <- cqgate(bits=c(i, 1),
              gate=sqgate(bit=1,
                          M=M, type=paste0("Uf", 2^(i-2)))) * x
```

```
  M <- M %*% M
}
x <- qft(x, inverse=TRUE, bits=c(2:(t+1)))

measurephi <- function(x, t) {
  xtmp <- measure(x)
  cbits <- genStateNumber(which(xtmp$value==1)-1, t+1)
  phi <- sum(cbits[1:t]/2^(1:t))
  return(invisible(phi))
}
phi <- measurephi(x, t=t)
2*pi*phi
```

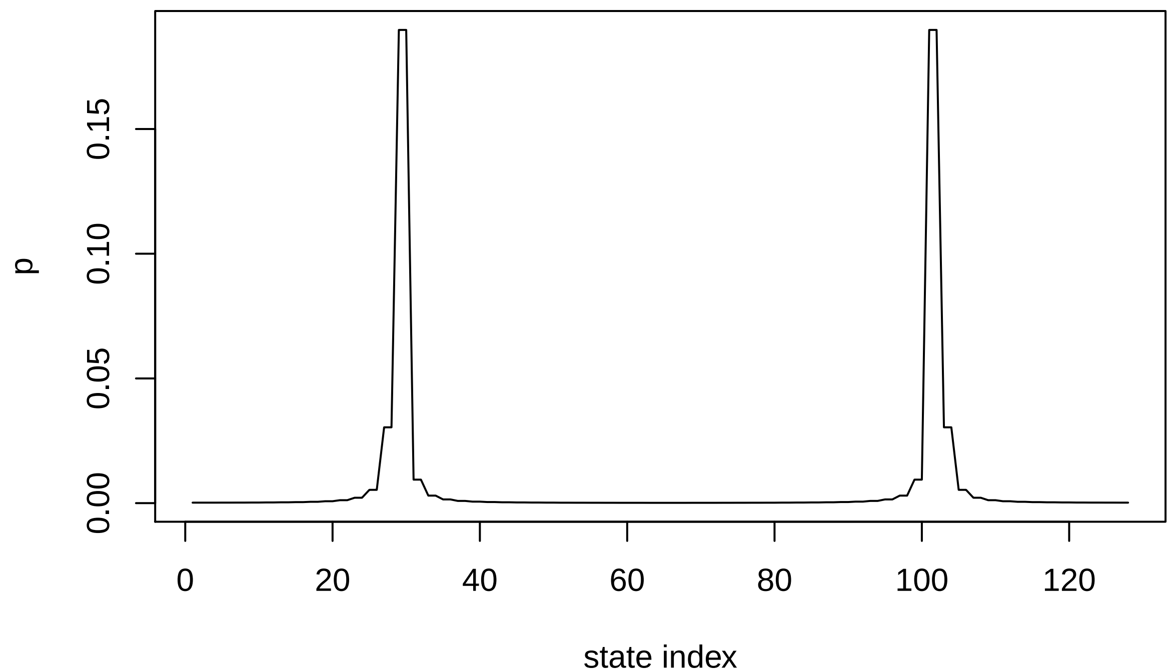
```
[1] 1.079922
```

```
phi-c(+alpha, 2*pi-alpha)/2/pi
```

```
[1] -0.04241071 -0.61383929
```

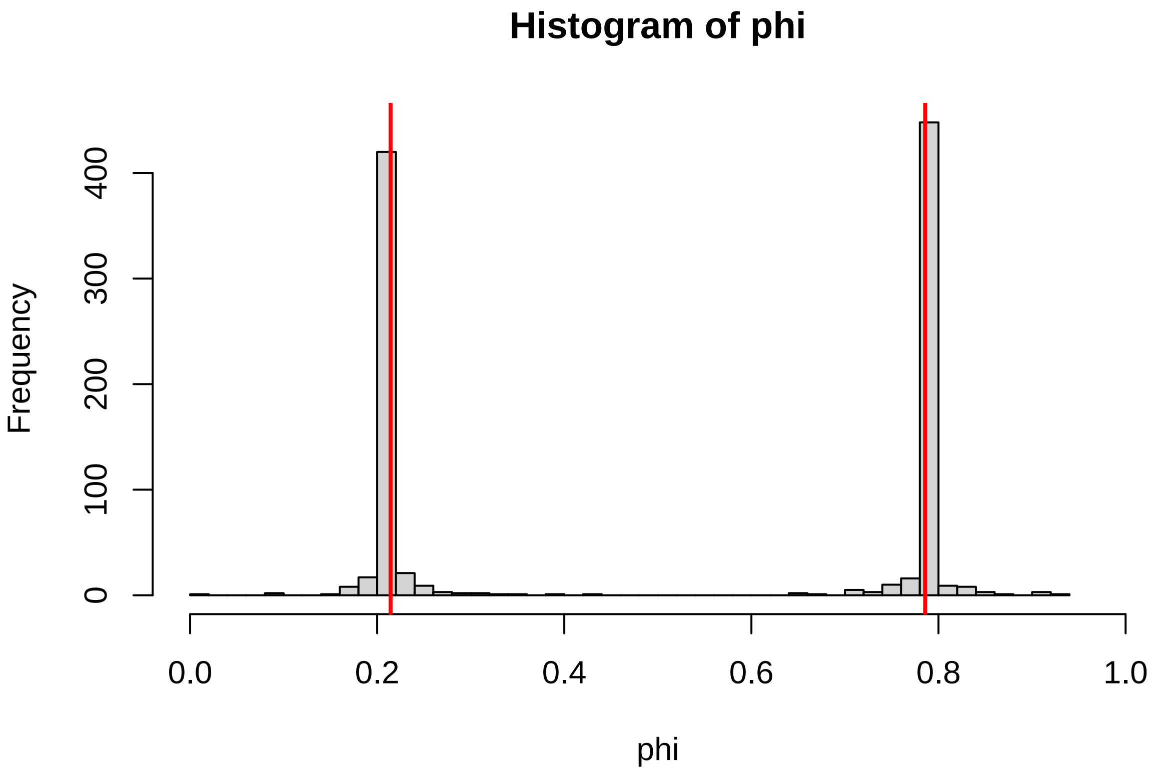
We can draw the probability distribution again and observe the two peaks corresponding to the two eigenvalues

```
plot(abs(x@coefs)^2, type="l",
     ylab="p", xlab="state index")
```



Let's measure 1000 times, which is easily possible in our simulator

```
phi <- c()
for(i in c(1:N)) {
  phi[i] <- measurephi(x, t)
}
hist(phi, breaks=2^t, xlim=c(0,1))
abline(v=c(alpha/2/pi, 1-alpha/2/pi), lwd=2, col="red")
```



The red vertical lines indicate the *true* values.