

# covMcd() – Considerations about Generalizing the FastMCD

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## 1 Introduction

The context is robust multivariate “location and scatter” estimation, which corresponds to estimating the first two moments in cases they exist. We assume data and a model

$$x_i \in \mathbb{R}^p, \quad i = 1, 2, \dots, n \quad (1)$$

$$x_i \sim \mathcal{F}(\mu, \Sigma), \quad \text{i.i.d.}; \quad \mu \in \mathbb{R}^p, \quad \Sigma \in \mathbb{R}^{p \times p}, \quad \text{positive definite}, \quad (2)$$

where a conceptual null model is the  $p$ -dimensional normal distribution. One typical assumption is that  $\mathcal{F}$  is a mixture with the majority component (“good data”) being  $\mathcal{N}_p(\mu, \Sigma)$  and other components modeling “the outliers”.

In other words, we want estimates  $(\hat{\mu}, \hat{\Sigma})$  which should be close to the true “good data”  $(\mu, \Sigma)$  — and do not say more here.

## 2 MCD and “the Fast” MCD (= fastmcd) Algorithm

The `robustbase` R package has featured a function `covMcd()` since early on (Feb. 2006) and that has been an interface to the Fortran routine provided by the original authors and (partly) described in [Rousseeuw and van Driessen \(1999\)](#). We describe shortly how the algorithm works, partly building on the documentation provided in the source (R, S, and Fortran) codes:

The minimum covariance determinant estimator of location and scatter (MCD) implemented in `covMcd()` is similar to R function `cov.mcd()` in `MASS`. The (“theoretical”) MCD looks for the  $h = h_\alpha (> 1/2)$  out of  $n$  observations whose classical covariance matrix has the lowest possible determinant. In more detail, we will use  $h = h_\alpha = h(\alpha, n, p) \approx \alpha \cdot (n + p + 1)$ , where as [Rousseeuw and van Driessen \(1999\)](#) mainly use (the default)  $\alpha = \frac{1}{2}$ , where  $h = h(1/2, n, p) = \lfloor \frac{n+p+1}{2} \rfloor$ . For general  $\alpha \geq \frac{1}{2}$ , the R implementation (derived from their original S code) uses  $h = h(\alpha, n, p) = \text{h.alpha.n}(\text{alpha}, n, p)$  (function in `robustbase`), which is

$$h = h_\alpha = h(\alpha, n, p) := \lfloor 2n_2 - n + 2\alpha(n - n_2) \rfloor, \quad \text{where } n_2 := \lfloor \frac{n + p + 1}{2} \rfloor. \quad (3)$$

The fraction  $\alpha \geq \frac{1}{2}$  can be chosen by the user, where  $\alpha = \frac{1}{2}$  is the most robust, and indeed,  $h_{1/2} = n_2 = \lfloor \frac{n+p+1}{2} \rfloor$ . Even in general, as long as  $n \gg p$ ,  $\alpha$  is approximately the *proportion* of the subsample size  $h$  in the full sample (size  $n$ ):

$$h \approx \alpha \cdot n \iff \alpha \approx \frac{h}{n}, \quad (4)$$

```
> require(robustbase)
> n <- c(5, 10, 20, 30, 50, 100, 200, 500)
> hmat <- function(alpha, p) cbind(n, h.alpha = h.alpha.n(alpha, n, p),
+   h. = floor(alpha * (n + p + 1)), alpha.n = round(alpha * n))
> hmat(alpha = 1/2, p = 3)
```

|      | n   | h.alpha | h.  | alpha.n |
|------|-----|---------|-----|---------|
| [1,] | 5   | 4       | 4   | 2       |
| [2,] | 10  | 7       | 7   | 5       |
| [3,] | 20  | 12      | 12  | 10      |
| [4,] | 30  | 17      | 17  | 15      |
| [5,] | 50  | 27      | 27  | 25      |
| [6,] | 100 | 52      | 52  | 50      |
| [7,] | 200 | 102     | 102 | 100     |
| [8,] | 500 | 252     | 252 | 250     |

```
> hmat(alpha = 3/4, p = 4)
```

|      | n   | h.alpha | h.  | alpha.n |
|------|-----|---------|-----|---------|
| [1,] | 5   | 5       | 7   | 4       |
| [2,] | 10  | 8       | 11  | 8       |
| [3,] | 20  | 16      | 18  | 15      |
| [4,] | 30  | 23      | 26  | 22      |
| [5,] | 50  | 38      | 41  | 38      |
| [6,] | 100 | 76      | 78  | 75      |
| [7,] | 200 | 151     | 153 | 150     |
| [8,] | 500 | 376     | 378 | 375     |

The breakdown point (for  $h > \frac{n}{2}$ ) then is

$$\epsilon_* = \frac{n - h + 1}{n}, \quad (5)$$

which is less than but close to  $\frac{1}{2}$  for  $\alpha = \frac{1}{2}$ , and in general,  $h/n \approx \alpha$ , the breakdown point is approximately,

$$\epsilon_* = \frac{n - h + 1}{n} \approx \frac{n - h}{n} = 1 - \frac{h}{n} \approx 1 - \alpha. \quad (6)$$

The raw MCD estimate of location, say  $\hat{\mu}_0$ , is then the average of these  $h$  points, whereas the raw MCD estimate of scatter,  $\hat{\Sigma}_0$ , is their covariance matrix, multiplied by a consistency factor `.MCDcons(p, h/n)` and (by default) a finite sample correction factor `.MCDcnp2(p, n, alpha)`, to make it consistent at the normal model and unbiased at small samples.

In practice, for reasonably sized  $n$ ,  $p$  and hence  $h$ , it is not feasible to search the full space of all  $\binom{n}{h}$   $h$ -subsets of  $n$  observations. Rather, the implementation of `covMcd` uses the Fast MCD algorithm of [Rousseeuw and van Driessen \(1999\)](#) to approximate the minimum covariance determinant estimator, see [Section 3](#).

Based on these raw MCD estimates,  $(\hat{\mu}_0, \hat{\Sigma}_0)$ , a reweighting step is performed, i.e.,  $\mathbf{V} \leftarrow \text{cov.wt}(\mathbf{x}, \mathbf{w})$ , where  $\mathbf{w}$  are weights determined by “outlyingness” with respect to the scaled raw MCD, using the “Mahalanobis”-like, robust distances  $d_i(\hat{\mu}_0, \hat{\Sigma}_0)$ , see [\(7\)](#). Again, a consistency factor and a finite sample correction factor are applied. The reweighted covariance is typically considerably more efficient than the raw one, see [Pison et al. \(2002\)](#).

The two rescaling factors for the reweighted estimates are returned in `cnp2`. Details for the computation of the finite sample correction factors can be found in [Pison et al. \(2002\)](#).

### 3 Fast MCD Algorithm – General notation

**Note:** In the following, apart from the mathematical notation, we also use variable names, e.g., `kmini`, used in the Fortran and sometimes R function code, in R package `robustbase`.

Instead of directly searching for  $h$ -subsets (among  $\binom{n}{h} \approx \binom{n}{n/2}$ ) the basic idea is to start with small subsets of size  $p+1$ , their center  $\mu$  and covariance matrix  $\Sigma$ , and a corresponding  $h$ -subset of the  $h$  observations with smallest (squared) (“Mahalanobis”-like) distances

$$d_i = d_i(\mu, \Sigma) := (x_i - \mu)' \Sigma^{-1} (x_i - \mu), \quad i = 1, 2, \dots, n, \quad (7)$$

and then use concentration steps (“C steps”) to (locally) improve the chosen set by iteratively computing  $\mu$ ,  $\Sigma$ , new distances  $d_i$  and a new set of size  $h$  with smallest distances  $d_i(\mu, \Sigma)$ . Each C step is proven to decrease the determinant  $\det(\Sigma)$  if  $\mu$  and  $\Sigma$  did change at all. Consequently, convergence to a local minimum is sure, as the number of  $h$ -subsets is finite.

To make the algorithm *fast* for non small sample size  $n$  the data set is split into “groups” or “sub-datasets” as soon as

$$n \geq 2n_0, \text{ where } n_0 := \mathbf{nmini} \text{ (} = 300, \text{ by default)}. \quad (8)$$

i.e., the default cutoff for “non small” is at  $n = 600$ . The *number* of such subsets in the original algorithm is maximally 5, and we now use

$$k_M = \mathbf{kmini} \text{ (} = 5, \text{ by default)}, \quad (9)$$

as upper limit. As above, we assume from now on that  $n \geq 2n_0$ , and let

$$k := \left\lfloor \frac{n}{n_0} \right\rfloor \geq 2 \quad (10)$$

and now distinguish the two cases,

$$\begin{cases} A. & k < k_M \iff n < k_M \cdot n_0 \\ B. & k \geq k_M \iff n \geq k_M \cdot n_0 \end{cases} \quad (11)$$

**In case A**  $k$  (=  $\mathbf{ngroup}$ ) subsets aka “groups” or “sub datasets” are used,  $k \in \{2, 3, \dots, k_M - 1\}$ , of group sizes  $n_j$ ,  $j = 1, \dots, k$  (see below). Note that case A may be empty because of  $2 \leq k < k_M$ , namely if  $k_M = 2$ . Hence, in case A, we have  $k_M \geq 3$ .

**in case B**  $k_M$  (=  $\mathbf{ngroup}$ ) groups each of size  $n_0$  are built and in the first stage, only a *subset* of  $k_M \cdot n_0 \leq n$  observations is used.

In both cases, the disjoint groups (“sub datasets”) are chosen at random from the  $n$  observations. For the group sizes for case A,  $n_j$ ,  $j = 1, \dots, k$ , we have

$$n_1 = \left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor \frac{n}{\left\lfloor \frac{n}{n_0} \right\rfloor} \right\rfloor \quad (\geq n_0) \quad (12)$$

$$n_j = n_1, \quad j = 2, \dots, j_* \quad (13)$$

$$n_j = n_1 + 1, \quad j = j_* + 1, \dots, k, \quad (14)$$

$$\text{where } j_* := k - r \in \{1, \dots, k\}, \quad (15)$$

$$\text{and } r := n - kn_1 = n - k \left\lfloor \frac{n}{k} \right\rfloor \in \{0, 1, \dots, k - 1\}, \quad (16)$$

where the range of  $j_*$ ,  $1, \dots, k$  in (15) is a consequence of the range of the integer division remainder  $r \in \{0, 1, \dots, k - 1\}$  in (16). Consequently, (14) maybe empty, namely iff  $r = 0$  ( $\iff n = k \cdot n_1$  is a multiple of  $k$ ):  $j_* = k$ , and all  $n_j \equiv n_1$ .

Considering the range of  $n_j$  in case A, the minimum  $n_1 \geq n_0$  in (12) is easy to verify. What is the maximal value of  $n_j$ , i.e., an upper bound for  $n_{\max} := n_1 + 1 \geq \max_j n_j$ ? Consider  $n_{1,\max}(k) = \max_{n, \text{ given } k} n_1 = \max_{n, \text{ given } k} \left\lfloor \frac{n}{k} \right\rfloor$ . Given  $k$ , the maximal  $n$  still fulfilling  $\left\lfloor \frac{n}{n_0} \right\rfloor = k$  is  $n = (k+1)n_0 - 1$  where  $\left\lfloor \frac{n}{n_0} \right\rfloor = k + \left\lfloor 1 - \frac{1}{n_0} \right\rfloor = k$ . Hence,  $n_{1,\max}(k) = \left\lfloor \frac{(k+1)n_0 - 1}{k} \right\rfloor = n_0 + \left\lfloor \frac{n_0 - 1}{k} \right\rfloor$ ,

and as  $k \geq 2$ , the maximum is at  $k = 2$ ,  $\max n_1 = \max_k n_{1,\max}(k) = n_0 + \lfloor \frac{n_0-1}{2} \rfloor = \lfloor \frac{3n_0-1}{2} \rfloor$ . Taken together, as  $n_j = n_1 + 1$  is possible, we have

$$\begin{aligned} n_0 \leq n_1 &\leq \left\lfloor \frac{3n_0 - 1}{2} \right\rfloor \\ n_0 \leq n_j &\leq \left\lfloor \frac{3n_0 + 1}{2} \right\rfloor, \quad j \geq 2. \end{aligned} \tag{17}$$

Note that indeed,  $\lfloor \frac{3n_0+1}{2} \rfloor$  is the length of the auxiliary vector `subndex` in the Fortran code.

## References

- Pison, G., S. Van Aelst, and G. Willems (2002). Small sample corrections for lts and mcd. *Metrika* 55(1-2), 111–123.
- Rousseeuw, P. J. and K. van Driessen (1999, August). A fast algorithm for the minimum covariance determinant estimator. *Technometrics* 41(3), 212–223.