

# Schwartz 1997 Two-Factor Model

## Technical Document

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### Abstract

The purpose of this document is to give the formulas and relations needed to understand the Schwartz two-factor commodity model (Schwartz, 1997). This includes parameter estimation using the Kalman filter, pricing of European options as well as computation of risk measures.

## 1 Introduction

This document describes the Schwartz two-factor model to the extent which is necessary to understand the R package *schwartz97*. The two factors are the spot price of a commodity together with its instantaneous convenience yield. It was introduced in Gibson and Schwartz (1990) and extended in Schwartz (1997) for the pricing of futures contracts. Miltersen and Schwartz (1998) and Hilliard and Reis (1998) presented equations for arbitrage free prices of European options on commodity futures. In what follows we fully rely on the above mentioned articles and state the corresponding formulas. In addition we derive the transition density of the two state variables.

## 2 Model

The spot price of the commodity and the instantaneous convenience yield are assumed to follow the joint stochastic process:

$$dS_t = (\mu - \delta_t)S_t dt + \sigma_S S_t dW_S \quad (1)$$

$$d\delta_t = \kappa(\alpha - \delta_t)dt + \sigma_\epsilon dW_\epsilon, \quad (2)$$

with Brownian motions  $W_S$  and  $W_\epsilon$  under the objective measure  $\mathbb{P}$  and correlation  $dW_S dW_\epsilon = \rho dt$ .

Under the pricing measure  $\mathbb{Q}$  the dynamics are of the form

$$dS_t = (r - \delta_t)S_t dt + \sigma_S S_t d\widetilde{W}_S \quad (3)$$

$$d\delta_t = [\kappa(\alpha - \delta_t) - \lambda]dt + \sigma_\epsilon d\widetilde{W}_\epsilon, \quad (4)$$

where the constant  $\lambda$  denotes the market price of convenience yield risk and  $\widetilde{W}_S$  and  $\widetilde{W}_\epsilon$  are  $\mathbb{Q}$ -Brownian motions. It may be handy to introduce a new mean-level for the convenience

yield process under  $\mathbb{Q}$

$$\tilde{\alpha} = \alpha - \lambda/\kappa, \quad (5)$$

which leads to the dynamics

$$d\delta_t = \kappa(\tilde{\alpha} - \delta_t)dt + \sigma_\epsilon d\widetilde{W}_\epsilon. \quad (6)$$

### 3 Distributions

#### 3.1 Joint Distribution of State Variables

The log-spot  $X_t = \log(S_t)$  and the convenience yield  $\delta_t$  are jointly normally distributed. The transition density is <sup>1</sup>

$$\begin{pmatrix} X_t \\ \delta_t \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} \mu_X(t) \\ \mu_\delta(t) \end{pmatrix}, \begin{pmatrix} \sigma_X^2(t) & \sigma_{X\delta}(t) \\ \sigma_{X\delta}(t) & \sigma_\delta^2(t) \end{pmatrix} \right), \quad (7)$$

with parameters

$$\mu_X(t) = X_0 + \left( \mu - \frac{1}{2}\sigma_S^2 - \alpha \right) t + (\alpha - \delta_0) \frac{1 - e^{-\kappa t}}{\kappa} \quad (8)$$

$$\mu_\delta(t) = e^{-\kappa t} \delta_0 + \alpha (1 - e^{-\kappa t}) \quad (9)$$

$$\sigma_X^2(t) = \frac{\sigma_\epsilon^2}{\kappa^2} \left( \frac{1}{2\kappa} (1 - e^{-2\kappa t}) - \frac{2}{\kappa} (1 - e^{-\kappa t}) + t \right) + 2 \frac{\sigma_S \sigma_\epsilon \rho}{\kappa} \left( \frac{1 - e^{-\kappa t}}{\kappa} - t \right) + \sigma_S^2 t \quad (10)$$

$$\sigma_\delta^2(t) = \frac{\sigma_\epsilon^2}{2\kappa} (1 - e^{-2\kappa t}) \quad (11)$$

$$\sigma_{X\delta}(t) = \frac{1}{\kappa} \left\{ \left( \sigma_S \sigma_\epsilon \rho - \frac{\sigma_\epsilon^2}{\kappa} \right) (1 - e^{-\kappa t}) + \frac{\sigma_\epsilon^2}{2\kappa} (1 - e^{-2\kappa t}) \right\}. \quad (12)$$

The mean-parameters given in (8) and (9) refer to the  $\mathbb{P}$ -dynamics. To obtain the parameters under  $\mathbb{Q}$  one can simply replace  $\mu$  by  $r$  and  $\alpha$  by  $\tilde{\alpha}$  defined in equation 5. Let the  $\mathbb{Q}$ -parameters be denoted by  $\tilde{\mu}_X(t)$  and  $\tilde{\mu}_\delta(t)$ .

### 4 Futures Price

It is worth to mention that the futures and forward price coincide since in our model the interest rate is assumed to be constant. In the rest of this document, the statements made about futures contracts therefore also hold for forward contracts.

Let the futures price at time  $t$  with time to maturity  $\tau = T - t$  be  $G(S_t, \delta_t, t, T)$ . For notational convenience we assume  $t = 0$  in what follows. At time zero the futures price is given by the  $\mathbb{Q}$ -expectation of  $S_T$ .

$$G(S_0, \delta_0, 0, T) = \mathbb{E}^{\mathbb{Q}}(S_T) = \exp \left\{ \tilde{\mu}_X(T) + \frac{1}{2} \sigma_X^2(T) \right\} \quad (13)$$

$$= S_0 e^{A(T) + B(T)\delta_0} \quad (14)$$

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<sup>1</sup>A derivation can be found in appendix A.

with

$$A(T) = \left( r - \tilde{\alpha} + \frac{1}{2} \frac{\sigma_\epsilon^2}{\kappa^2} - \frac{\sigma_S \sigma_\epsilon \rho}{\kappa} \right) T + \frac{1}{4} \sigma_\epsilon^2 \frac{1 - e^{-2\kappa T}}{\kappa^3} + \left( \kappa \tilde{\alpha} + \sigma_S \sigma_\epsilon \rho - \frac{\sigma_\epsilon^2}{\kappa} \right) \frac{1 - e^{-\kappa T}}{\kappa^2}, \quad (15)$$

$$B(T) = -\frac{1 - e^{-\kappa T}}{\kappa}. \quad (16)$$

#### 4.1 Distribution of Futures Prices

According to (13) the futures price follows a log-normal law. That is, at time zero the  $T$ -futures price at time  $t$  has the following distribution under  $\mathbb{Q}^2$ :

$$\log G(S_t, \delta_t, t, T) \sim \mathcal{N}(\mu_G(t, T), \sigma_G^2(t, T)), \quad (17)$$

where

$$\mu_G(t, T) = \tilde{\mu}_X(t) + A(T - t) + B(T - t)\tilde{\mu}_\delta(t) \quad (18)$$

$$\sigma_G^2(t, T) = \sigma_X^2(t) + 2B(T - t)\sigma_{X\delta}(t) + B(T - t)^2\sigma_\delta(t). \quad (19)$$

## 5 European Commodity Options

The fair price of a European call option on a commodity futures contract was derived as a special case of more general models in Miltersen and Schwartz (1998) and Hilliard and Reis (1998).

Here we give the formula for the two-factor model. In this setting, the price of a European call option  $C$  at time zero with maturity  $t$ , exercise price  $K$  written on a commodity futures contract with maturity  $T$  is given by

$$C^G = \mathbb{E}^{\mathbb{Q}} [e^{-rt}(G(S_t, \delta_t, t, T) - K)^+] \quad (20)$$

Since the futures price  $G(S_t, \delta_t, t, T)$  is log-normally distributed we obtain a Black-Scholes type formula for the call price  $C^G$ .

$$C^G = P(0, t) \{G(0, T)\Phi(d_+) - K\Phi(d_-)\} \quad (21)$$

with

$$\begin{aligned} d_\pm &= \frac{\log \frac{G(0, T)}{K} \pm \frac{1}{2}\sigma^2}{\sigma} \\ \sigma^2 &= \sigma_S^2 t + \frac{2\sigma_S \sigma_\epsilon \rho}{\kappa} \left( \frac{1}{\kappa} e^{-\kappa T} (e^{\kappa t} - 1) - t \right) \\ &\quad + \frac{\sigma_\epsilon^2}{\kappa^2} \left( t + \frac{1}{2\kappa} e^{-2\kappa T} (e^{2\kappa t} - 1) - \frac{2}{\kappa} e^{-\kappa T} (e^{\kappa t} - 1) \right) \end{aligned}$$

and  $\Phi$  being the standard Gaussian distribution function.

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<sup>2</sup>Note that the  $\mathbb{Q}$ -dynamics is primarily interesting to value derivatives on the futures price. For simulation studies and dynamic financial analysis the real-world ( $\mathbb{P}$ ) dynamics is of relevance. To get the  $\mathbb{P}$ -dynamics all “tilde-parameters” have to be replaced by the ones without tilde.

The following put-call parity is established

$$C^G - P^G = P(0, t) \{G(0, t) - K\}. \quad (22)$$

Thus, the price for a European put option  $P^G$  at time zero with maturity  $t$ , exercise price  $K$  written on a commodity futures contract with maturity  $T$  becomes

$$P^G = P(0, t) \{K\Phi(-d_-) - G(0, T)\Phi(-d_+)\}. \quad (23)$$

Remark: For the special case when the exercise time  $t$  of the option and the maturity  $T$  of the futures contract coincide, formulas (21) and (23) still hold. However, the options we price for  $t = T$  are no longer options written on futures contracts with maturity  $T$  but rather options with exercise time  $T$ , written on a commodity *spot* contract.

## 6 Parameter Estimation

This section demonstrates an elegant way of estimating the Schwartz two-factor model. That is estimating the model parameters using the Kalman filter as in Schwartz (1997). Subsection 6.1 shows how the Schwartz two-factor model can be expressed in state space form. Once the model has been cast in this form the likelihood can be computed and numerically maximized.

### 6.1 State Space Representation

Let  $\mathbf{y}_t$  denote a  $(n \times 1)$  vector of futures prices observed at date  $t$  and  $\boldsymbol{\alpha}_t$  denote the  $(2 \times 1)$  *state vector* of the spot price and the convenience yield. The state space representation of the dynamics of  $\mathbf{y}$  is given by the linear system of equations

$$\mathbf{y}_t = \mathbf{c}_t + Z_t \boldsymbol{\alpha}_t + G_t \boldsymbol{\eta}_t \quad (24)$$

$$\boldsymbol{\alpha}_{t+1} = \mathbf{d}_t + T_t \boldsymbol{\alpha}_t + H_t \boldsymbol{\epsilon}_t, \quad (25)$$

where  $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, I_2)$  and  $\boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, I_n)$ .  $G_t$  and  $H_t$  are assumed to be time-invariant. The errors (“innovations”) in the measurement equation (24) are further assumed to be independent in the implementation of this package

$$\mathbf{G}_t \mathbf{G}'_t = \begin{pmatrix} g_{11}^2 & & \\ & \ddots & \\ & & g_{nn}^2 \end{pmatrix}. \quad (26)$$

Using the functions  $A(\cdot)$  and  $B(\cdot)$  defined in (15) and (16) the components of the state space representation (24) and (25) are

$$\boldsymbol{\alpha}_{t+\Delta t} = \begin{pmatrix} X_{t+\Delta t} \\ \delta_{t+\Delta t} \end{pmatrix} \quad (27)$$

$$\mathbf{T}_t = \begin{pmatrix} 1 & \frac{1}{\kappa} (e^{-\kappa \Delta t} - 1) \\ 0 & e^{-\kappa \Delta t} \end{pmatrix} \quad (28)$$

$$\mathbf{d}_t = \begin{pmatrix} (\mu - \frac{1}{2} \sigma_S^2 - \alpha) \Delta t + \frac{\alpha}{\kappa} (1 - e^{-\kappa \Delta t}) \\ \alpha (1 - e^{-\kappa \Delta t}) \end{pmatrix} \quad (29)$$

$$\mathbf{H}_t \mathbf{H}'_t = \begin{pmatrix} \sigma_X^2(\Delta t) & \sigma_{X\delta}(\Delta t) \\ \sigma_{X\delta}(\Delta t) & \sigma_\delta^2(\Delta t) \end{pmatrix} \quad (30)$$

$$\mathbf{y}_t = \begin{pmatrix} \log G_t(1) \\ \vdots \\ \log G_t(n) \end{pmatrix} \quad \mathbf{Z}_t = \begin{pmatrix} 1 & B(m_t(1)) \\ \vdots & \vdots \\ 1 & B(m_t(n)) \end{pmatrix} \quad (31)$$

$$\mathbf{c}_t = \begin{pmatrix} A(m_t(1)) \\ \vdots \\ A(m_t(n)) \end{pmatrix} \quad \mathbf{G}_t \mathbf{G}'_t = \begin{pmatrix} g_{11}^2 & & \\ & \ddots & \\ & & g_{nn}^2 \end{pmatrix} \quad (32)$$

where  $X_t = \log S_t$ ,  $\Delta t = t_{k+1} - t_k$  and  $m_t(i)$  denotes the remaining time to maturity of the  $i$ -th closest to maturity futures  $G_t(i)$ .

## A Derivation of the joint distribution

The joint dynamics of the commodity log-price  $X_t = \log S_t$  and the spot convenience yield  $\delta_t$  reads in an orthogonal decomposition of (1) and (2)

$$dX_t = \left( \mu - \delta_t - \frac{1}{2}\sigma_S^2 \right) dt + \sigma_S \sqrt{1 - \rho^2} dW_t^1 + \sigma_S \rho dW_t^2 \quad (33)$$

$$d\delta_t = \kappa(\alpha - \delta_t)dt + \sigma_\epsilon dW_t^2, \quad (34)$$

Equation (34) can be solved using the substitution  $\tilde{\delta}_t = e^{\kappa t} \delta_t$  and Ito's lemma.

$$\delta_t = e^{-\kappa t} \delta_0 + \alpha (1 - e^{-\kappa t}) + \sigma_\epsilon e^{-\kappa t} \int_0^t e^{\kappa u} dW_u^2 \quad (35)$$

Plugging (35) into (33) gives

$$X_t = X_0 + \int_0^t dX_u \quad (36)$$

$$= X_0 + \left( \mu - \frac{1}{2}\sigma_S^2 \right) t - \int_0^t \delta_u du + \int_0^t \sigma_S \sqrt{1 - \rho^2} dW_u^1 + \int_0^t \sigma_S \rho dW_u^2. \quad (37)$$

Let's have a look at the integral  $\int_0^t \delta_u du$ .

$$\int_0^t \delta_u du = \int_0^t e^{-\kappa u} \delta_0 du + \int_0^t \alpha (1 - e^{-\kappa u}) du + \int_0^t \sigma_\epsilon e^{-\kappa u} \left( \int_0^u e^{\kappa s} dW_s^2 \right) du \quad (38)$$

For the integral  $\int_0^t e^{-\kappa u} \left( \int_0^u e^{\kappa s} dW_s^2 \right) du$  we use Fubini's theorem to interchange the order of integration:

$$\int_0^t \left( \int_0^u e^{-\kappa u} e^{\kappa s} dW_s^2 \right) du = \int_0^t \left( \int_s^t e^{-\kappa u} e^{\kappa s} du \right) dW_s^2 \quad (39)$$

$$= \int_0^t \frac{1}{\kappa} \left( 1 - e^{-\kappa(t-s)} \right) dW_s^2 \quad (40)$$

Plugging eq. (40) into eq. (38) and solving the Riemann integrals yields

$$\int_0^t \delta_u du = \frac{\delta_0}{\kappa} (1 - e^{-\kappa t}) + \alpha t - \frac{\alpha}{\kappa} (1 - e^{-\kappa t}) + \sigma_\epsilon \int_0^t \frac{1}{\kappa} \left( 1 - e^{-\kappa(t-s)} \right) dW_s^2. \quad (41)$$

This leaves us with the following expression for  $X_t$ :

$$\begin{aligned} X_t = & X_0 + \left( \mu - \frac{1}{2}\sigma_S^2 - \alpha \right) t + (\alpha - \delta_0) \frac{1 - e^{-\kappa t}}{\kappa} + \\ & + \int_0^t \sigma_S \sqrt{1 - \rho^2} dW_u^1 + \int_0^t \left\{ \sigma_S \rho + \frac{\sigma_\epsilon}{\kappa} \left( e^{-\kappa(t-u)} - 1 \right) \right\} dW_u^2. \end{aligned} \quad (42)$$

The log-spot  $X_t$  and the convenience yield  $\delta_t$  are jointly normally distributed with expectations

$$E(X_t) = \mu_X = X_0 + \left( \mu - \frac{1}{2}\sigma_S^2 - \alpha \right) t + (\alpha - \delta_0) \frac{1 - e^{-\kappa t}}{\kappa} \quad (43)$$

$$E(\delta_t) = \mu_\delta = e^{-\kappa t} \delta_0 + \alpha (1 - e^{-\kappa t}). \quad (44)$$

The variance are obtained using expectation rules for Ito integrals and the Ito isometry.

$$\text{var}(X_t) = \sigma_X^2 = \frac{\sigma_\epsilon^2}{\kappa^2} \left\{ \frac{1}{2\kappa} (1 - e^{-2\kappa t}) - \frac{2}{\kappa} (1 - e^{-\kappa t}) + t \right\} + 2 \frac{\sigma_S \sigma_\epsilon \rho}{\kappa} \left( \frac{1 - e^{-\kappa t}}{\kappa} - t \right) + \sigma_S^2 t \quad (45)$$

$$\text{var}(\delta_t) = \sigma_\delta^2 = \frac{\sigma_\epsilon^2}{2\kappa} (1 - e^{-2\kappa t}) \quad (46)$$

$$\text{cov}(X_t, \delta_t) = \sigma_{X\delta} = \frac{1}{\kappa} \left[ \left( \sigma_S \sigma_\epsilon \rho - \frac{\sigma_\epsilon^2}{\kappa} \right) (1 - e^{-\kappa t}) + \frac{\sigma_\epsilon^2}{2\kappa} (1 - e^{-2\kappa t}) \right] \quad (47)$$

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