1 Introduction

1.1 Why the POT package?

The POT package is an add-on package for the R statistical software (R Development Core Team 2006). The main goal of this package is to develop tools to perform statistical analyses of Peaks Over a Threshold (POT).

Most of functions are related to the Extreme Value Theory (EVT). Coles (2001) gives a comprehensive introduction to the EVT, while Kluppelberg and Mikosch (1997) present advanced results.

1.2 Obtaining the package/guide

The package can be downloaded from CRAN (The Comprehensive R Archive Network) at http://cran.r-project.org/. This guide (in pdf) will be in the directory POT/doc/ underneath wherever the package is installed.

1.3 Contents

To help users to use properly the POT package, this guide contains practical examples on the use of this package. Section 2 introduce quickly the Extreme Value Theory (EVT). Some basic examples are described in section 3, while section 4 gives a concrete statistical analysis of extreme value for river Adièrèes at Beaujeu (FRANCE).

1.4 Citing the package/guide

To cite this guide or the package in publications please use the following bibliographic database entry.

```
title = {A User’s Guide to the POT Package (Version 1.0)},
author = {Ribatet, M. A.},
year = {2006},
month = {August},
url = {http://cran.r-project.org/}
```

1.5 Caveat

I have checked these functions as best I can but, as ever, they may contain bugs. If you find a bug or suspected bug in the code or the documentation please report it to me at ribatet@hotmail.com. Please include an appropriate subject line.
1.6 Legalese

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2 An Introduction to the EVT

2.1 The univariate case

Even if this package is only related to peaks over a threshold, a classical introduction to the EVT must deal with “block maxima”. Let $X_1, \ldots, X_n$ be a series of independent and identically distributed random variables with common distribution function $F$. Let $M_n = \max(X_1, \ldots, X_n)$.

Suppose there exists normalizing constants $a_n > 0$ and $b_n$, such that:

$$
\Pr \left[ \frac{M_n - b_n}{a_n} \leq y \right] = F^n(a_n y + b_n) \longrightarrow G(y), \quad n \to +\infty
$$

(2.1)

for all $y \in \mathbb{R}$, where $G$ is a non-degenerate distribution function. According to the Extremal Types Theorem [Fisher and Tippett, 1928], $G$ must be either Fréchet, Gumbel or negative Weibull. Jenkinson [1955] noted that these three distributions can be merged into a single parametric family: the Generalized Extreme Value (GEV) distribution. The GEV has a distribution function defined by:

$$
G(y) = \exp \left[ - \left( 1 + \xi \frac{y - \mu}{\sigma} \right)^{-1/\xi} \right],
$$

(2.2)

where $(\mu, \sigma, \xi)$ are the location, scale and shape parameters respectively, $\sigma > 0$ and $z_+ = \max(z, 0)$.

The Fréchet case is obtained when $\xi > 0$, the negative Weibull when $\xi < 0$ while the Gumbel case is defined by continuity when $\xi \to 0$.

From this result, Pickands [1975] showed that the limiting distribution of normalized excesses of a threshold $\mu$ as the threshold approaches the endpoint $\mu_{\text{end}}$ of the variable of interest is the Generalized Pareto Distribution (GPD). That is, if $X$ is a random variable which holds (2.1), then:

$$
\Pr [X \leq y | X > \mu] \longrightarrow H(y), \quad \mu \to \mu_{\text{end}}
$$

(2.3)

with

$$
H(y) = 1 - \left( 1 + \xi \frac{y - \mu}{\sigma} \right)^{-1/\xi},
$$

(2.4)

where $(\mu, \sigma, \xi)$ are the location, scale and shape parameters respectively, $\sigma > 0$ and $z_+ = \max(z, 0)$. Note that the Exponential distribution is obtained by continuity as $\xi \to 0$.

In practice, these two asymptotic results motivated modelling block maxima with a GEV, while peaks over threshold with a GPD.

2.2 The multivariate case

When dealing with multivariate extremes, it is usual to transform data to a particular distribution. For example, Falk and Reiss [2005] used the inverted standard exponential distribution – $\Pr[Z \leq z] = \exp(z)$,
where \( F_j \) is the distribution of the \( j \)-th margin.

Obviously, in practice, the margins \( F_j \) are unknown. When dealing with extremes, the univariate EVT tells us what to do. Thus, if block maxima or peaks over a threshold are of interest, we must replace \( F_j \) with GEV or GPD respectively.

**Definition 2.2.1.** A multivariate extreme value distribution in dimension \( d \) has representation:

\[
G(y_1, \ldots, y_d) = \exp \left[ -V(z_1, \ldots, z_d) \right]
\]

with

\[
V(z_1, \ldots, z_d) = \int_{T_p} \max_{j=1, \ldots, d} \left( \frac{q_j}{z_j} \right) dH(q_1, \ldots, q_d)
\]

where \( H \) is a measure with mass 2 called *spectral density* defined on the set

\[
T_p = \left\{ (q_1, \ldots, q_d) : q_j \geq 0, \sum_{j=1}^{d} q_j^2 = 1 \right\}
\]

with the constraint

\[
\int_{T_p} q_j dH(q_j) = 1, \quad \forall j \in \{1, \ldots, d\}
\]

The \( V \) function is often called *exponential measure* ([Krüppelberg and May, 2006]) and is an homogeneous function of order \(-1\).

Contrary to the univariate case, there is an infinity of functions \( V \) for \( d > 1 \). Thus, it is usual to used specific parametric families for \( V \). Several examples for these families are given in Annexe A.

Another representation for a multivariate extreme value distribution is the Pickands’ representation ([Pickands, 1981]). We give here only the bivariate case.

**Definition 2.2.2.** A bivariate extreme value distribution has the Pickands’ representation:

\[
G(y_1, y_2) = \exp \left[ -\left( \frac{1}{z_1} + \frac{1}{z_2} \right) A \left( \frac{z_2}{z_1 + z_2} \right) \right]
\]

with

\[
A : [0, 1] \rightarrow [0, 1]
\]

\[
w \mapsto A(w) = \int_0^1 \max \{w (1 - q), (1 - w) q\} dH(q)
\]

In particular, the functions \( V \) and \( A \) are linked by the relation:

\[
A(w) = \frac{V(z_1, z_2)}{z_1^{-1} + z_2^{-1}}, \quad w = \frac{z_2}{z_1 + z_2}
\]

The dependence function \( A \) holds:

1. \( A(0) = A(1) = 1; \)
2. \( \max(w, 1 - w) \leq A(w) \leq 1, \forall w; \)
3. \( A \) is convex;
4. Two random variables (with unit Fréchet margins) are independent if \( A(w) = 1, \forall w; \)
5. Two random variables (with unit Fréchet margins) are perfectly dependent if $A(w) = \max(w, 1-w), \forall w$.

We define the multivariate extreme value distributions which are identical to the block maxima approach in higher dimensions. We now establish the multivariate theory for peaks over threshold.

According to Resnick (1987, Prop. 5.15), multivariate peaks over thresholds $u_j$ has the same representation than for block maxima. Only the margins $F_j$ must be replaced by GPD instead of GEV. Thus,

$$F(y_1, \ldots, y_d) = \exp\left(-V\left(-\frac{1}{\log F_1(y_1)}, \ldots, -\frac{1}{\log F_d(y_d)}\right)\right), \quad y_j > u_j \tag{2.7}$$

3 Basic Use

3.1 Random Numbers and Distribution Functions

First of all, let's start with basic stuffs. The POT package uses the R convention for random numbers generation and distribution function features.

```r
> library(POT)
> rgpd(5, loc = 1, scale = 2, shape = -0.2)
[1] 1.775989 2.388942 5.265147 1.513448 1.364476
> rgpd(6, c(1, -5), 2, -0.2)
> rgpd(6, 0, c(2, 3), 0)
[1] 1.7915813 3.4027201 2.9702331 0.5448282 3.8591534 2.2262692
> pgpd(c(9, 15, 20), 1, 2, 0.25)
[1] 0.9375000 0.9825149 0.9922927
> qgpd(c(.25, .5, .75), 1, 2, 0)
[1] 1.575364 2.386294 3.772589
> dgpd(c(9, 15, 20), 1, 2, 0.25)
[1] 0.015625000 0.003179117 0.001141829
```

Several options can be passed to three of these four functions. In particular:

- for “pgpd”, user can specify if non exceedence or exceedence probability should be computed with option `lower.tail = TRUE` or `lower.tail = FALSE` respectively;
- for “qgpd”, user can specify if quantile is related to non exceedence or exceedence probability with option `lower.tail = TRUE` or `lower.tail = FALSE` respectively;
- for “dgpd”, user can specify if the density or the log-density should be computed with option `log = FALSE` or `log = TRUE` respectively.
3.2 Threshold Selection

The location for the GPD or equivalently the threshold is a particular parameter as must often it is not
estimated as the other ones. All methods to define a suitable threshold use the asymptotic approximation
defined by equation (2.3). In other words, we select a threshold for which the asymptotic distribution \( H \)
in equation (2.4) is a good approximation.

The POT package has several tools to define a reasonable threshold. For this purpose, the user must use
tcplot, mrlplot, lsmomplot, exiplot and diplot functions.

The main goal of threshold selection is to selects enough events to reduce the variance; but not too much
as we could select events coming from the central part of the distribution\(^1\) and induce bias.

### 3.2.1 Threshold Choice plot: \( \text{tcplot} \)

Let \( X \sim GP(\mu_0, \sigma_0, \xi_0) \). Let \( \mu_1 \) be another threshold as \( \mu_1 > \mu_0 \). The random variable \( X|X > \mu_1 \)
is also GPD with updated parameters \( \sigma_1 = \sigma_0 + \xi_0(\mu_1 - \mu_0) \) and \( \xi_1 = \xi_0 \). Let
\[
\sigma_* = \sigma_1 - \xi_1 \mu_1
\]
(3.1)

With this new parametrization, \( \sigma_* \) is independent of \( \mu_1 \). Thus, estimates of \( \sigma_* \) and \( \xi_1 \) are constant for
all \( \mu_1 > \mu_0 \) if \( \mu_0 \) is a suitable threshold for the asymptotic approximation.

Threshold choice plots represent the points defined by:

\[
\{(\mu_1, \sigma_*): \mu_1 \leq x_{\text{max}}\} \quad \text{and} \quad \{(\mu_1, \xi_1): \mu_1 \leq x_{\text{max}}\}
\]
(3.2)

where \( x_{\text{max}} \) is the maximum of the observations \( x \).

Moreover, confidence intervals can be computed using Fisher information.

Here is an application.

```r
> x <- runif(10000)
> par(mfrow=c(1,2))
> tcplot(x, u.range = c(0.9, 0.995))
```

Results of the \( \text{tcplot} \) function is displayed in Figure \[1\]. We can see clearly that a threshold around 0.98
is a reasonable choice. However, in practice decision are not so clear-cut as for this synthetic example.

### 3.2.2 Mean Residual Life Plot: \( \text{mrlplot} \)

The mean residual life plot is based on the theoretical mean of the GPD. Let \( X \) be a r.v. distributed
as \( GPD(\mu, \sigma, \xi) \). Then, theoretically we have:

\[
E[X] = \mu + \frac{\sigma}{1 - \xi^*}, \quad \text{for } \xi < 1
\]
(3.3)

When \( \xi \geq 1 \), the theoretical mean is infinite.

In practice, if \( X \) represents excess over a threshold \( \mu_0 \), and if the approximation by a GPD is good enough,
we have:

\[
E[X - \mu_0|X > \mu_0] = \frac{\sigma \mu_0}{1 - \xi}
\]
(3.4)

For all new threshold \( \mu_1 \) such as \( \mu_1 > \mu_0 \), excesses above the new threshold are also approximate by a
GPD with updated parameters - see section 3.2.1 Thus,

\[
E[X - \mu_1|X > \mu_1] = \frac{\sigma \mu_1}{1 - \xi} = \frac{\sigma \mu_0 + \xi \mu_1}{1 - \xi}
\]
(3.5)

\(^1\)i.e. not extreme events.
Figure 1: The threshold selection using the tcplot function
The quantity \(\mathbb{E}[X - \mu_1 | X > \mu_1]\) is linear in \(\mu_1\). Or, \(\mathbb{E}[X - \mu_1 | X > \mu_1]\) is simply the mean of excesses above the threshold \(\mu_1\) which can easily be estimated using the empirical mean.

A mean residual life plot consists in representing points:

\[
\left\{ \left( \mu, \frac{1}{n_\mu} \sum_{i=1}^{n_\mu} x_{i,n_\mu} - \mu \right) : \mu \leq x_{\text{max}} \right\}
\]

where \(n_\mu\) is the number of observations \(x\) above the threshold \(\mu\), \(x_{i,n_\mu}\) is the \(i\)-th observation above the threshold \(\mu\) and \(x_{\text{max}}\) is the maximum of the observations \(x\).

Confidence intervals can be added to this plot as the empirical mean can be supposed to be normally distributed (Central Limit Theorem). However, normality doesn’t hold anymore for high threshold as there are less and less excesses. Moreover, by construction, this plot always converge to the point \((x_{\text{max}}, 0)\).

Here is another synthetic example.

```r
> x <- rnorm(10000)
> mrlplot(x, u.range = c(1, quantile(x, probs = 0.995)),
+ col = c("green", "black", "green"), nt = 200)
```

Figure 2 displays the mean residual life plot. A threshold around 2.5 should be reasonable.
3.2.3 L-Moments plot: *lmomplot*

L-moments are summary statistics for probability distributions and data samples. They are analogous to ordinary moments – they provide measures of location, dispersion, skewness, kurtosis, and other aspects of the shape of probability distributions or data samples – but are computed from linear combinations of the ordered data values (hence the prefix L).

For the GPD, the following relation holds:

\[
\tau_4 = \frac{\tau_3}{\tau_3 + 5} \tag{3.7}
\]

where \(\tau_4\) is the **L-Kurtosis** and \(\tau_3\) is the **L-Skewness**.

The **L-Moment** plot represents points defined by:

\[
\{(\hat{\tau}_3, u, \hat{\tau}_4, u) : u \leq x_{\text{max}}\} \tag{3.8}
\]

where \(\hat{\tau}_3\) and \(\hat{\tau}_4\) are estimations of the L-Kurtosis and L-Skewness based on excesses over threshold \(u\) and \(x_{\text{max}}\) is the maximum of the observations \(x\). The theoretical curve defined by equation (3.7) is traced as a guideline.

Here is a trivial example.

```r
> x <- c(1 - abs(rnorm(200, 0, 0.2)), rgpd(100, 1, 2, 0.25))
> lmomplot(x, u.range = c(0.9, quantile(x, probs = 0.9)), identify = FALSE)
```

Figure 3.2.3 displays the L-Moment plot. By passing option `identify = TRUE` user can click on the graphic to identify the threshold related to the point selected.

We found that this graphic has often poor performance on real data.

3.2.4 Dispersion Index Plot: *diplot*

The Dispersion Index plot is particularly useful when dealing with time series. The EVT states that excesses over a threshold can be approximated by a GPD. However, the EVT also states that the occurrences of these excesses must be represented by a Poisson process.

Let \(X\) be a r.v. distributed as a Poisson distribution with parameter \(\lambda\). That is:

\[
\Pr [X = k] = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{N}. \tag{3.9}
\]

Thus, we have \(\mathbb{E}[X] = \text{Var}[X]\). [Cunnane (1979)] introduced a Dispersion Index statistic defined by:

\[
DI = \frac{s^2}{\lambda} \tag{3.10}
\]

where \(s^2\) is the intensity of the Poisson process and \(\lambda\) the mean number of events in a block - most often this is a year. Moreover, a confidence interval can be computed by using a \(\chi^2\) test:

\[
\alpha = \left[\frac{\chi^2_{1-(\lambda-\alpha)/2, M-1}}{M}, \frac{\chi^2_{1-(\lambda-\alpha)/2, M-1}}{M-1}\right] \tag{3.11}
\]

where \(\Pr [DI \in I_\alpha] = \alpha\).

For the next example, we use the data set `ardieres` included in the POT package. Moreover, as `ardieres` is a time series, and thus strongly auto-correlated, we must “extract” extreme events while preserving independence between events. This is achieved using function `clust`.

```r
> data(ardieres)
> events <- clust(ardieres, u = 2, tim.cond = 8 / 365, clust.max = TRUE)
> diplot(events, u.range = c(2, 20))
```

The Dispersion Index plot is presented in Figure 4. From this figure, a threshold around 5 should be reasonable.

\(^2\)The clust function will be presented later in section 3.6.
Figure 3: The threshold selection using the lmomplot function
Figure 4: The threshold selection using the diplot function
3.3 Fitting the GPD

3.3.1 The univariate case

The main function to fit the GPD is called `fitgpd`. This is a generic function which can fit the GPD according to several estimators. There are currently 17 estimators available: method of moments `moments`, maximum likelihood `mle`, biased and unbiased probability weighted moments `pwmb`, `pwmu`, mean power density divergence `mdpd`, median `med`, pickands' `pickands`, maximum penalized likelihood `mple` and maximum goodness-of-fit `mgf` estimators. For the `mgf` estimator, the user has to select which goodness-of-fit statistics must be used. These statistics are the Kolmogorov-Smirnov, Cramer von Mises, Anderson Darling and modified Anderson Darling. See the html help page of the `fitgpd` function to see all of them. Details for these estimators can be found in Coles (2001), Hosking and Wallis (1987), Juárez and Schucany (2004), Peng and Welsh (2001) and Pickands (1975).

The MLE is a particular case as it is the only one which allows varying threshold. Moreover, two types of standard errors are available: “expected” or “observed” information of Fisher. The option `obs.fish` specifies if we want observed (`obs.fish = TRUE`) or expected (`obs.fish = FALSE`).

As Pickands’ estimator is not always feasible, user must check the message of feasibility return by function `fitgpd`.

We give here several didactic examples.

```r
> x <- rgpd(200, 1, 2, 0.25)
> mom <- fitgpd(x, 1, "moments")$param
> mle <- fitgpd(x, 1, "mle")$param
> pwmu <- fitgpd(x, 1, "pwmu")$param
> pwmb <- fitgpd(x, 1, "pwmb")$param
> pickands <- fitgpd(x, 1, "pickands")$param
> med <- fitgpd(x, 1, "med", start = list(scale = 2, shape = 0.25))$param
> mdpd <- fitgpd(x, 1, "mdpd")$param
> mple <- fitgpd(x, 1, "mple")$param
> ad2r <- fitgpd(x, 1, "mgf", stat = "AD2R")$param
> print(rbind(mom, mle, pwmu, pwmb, pickands, med, mdpd, mple, ad2r))

 scale shape
 mom 2.393164 0.1850449
 mle 2.231046 0.2471458
 pwmu 2.205225 0.2490445
 pwmb 2.217459 0.2448786
 pickands 2.017325 0.3185735
 med 2.070678 0.2507760
 mdpd 2.200819 0.2654388
 mple 2.257466 0.2329870
 ad2r 2.322558 0.2260157
```

The MLE, MPLE and MGF estimators allow to fix either the scale or the shape parameter. For example, if we want to fit a Exponential distribution, just do (with eventually a fixed scale parameter):

```r
> x <- rgpd(100, 1, 2, 0)
> fitgpd(x, thresh = 1, shape = 0, est = "mle")
```

Estimator: MLE
Deviance: 344.1673
AIC: 346.1673

Varying Threshold: FALSE
Threshold Call: 1
Number Above: 100
Proportion Above: 1

Estimates
scale
2.056

Standard Error Type: observed

Standard Errors
scale
0.2056

Asymptotic Variance Covariance
  scale
  scale 0.04228

Optimization Information
  Convergence: successful
  Function Evaluations: 11
  Gradient Evaluations: 1

> fitgpd(x, thresh = 1, scale = 2, est = "mle")

Estimator: MLE
Deviance: 343.7133
  AIC: 345.7133

Varying Threshold: FALSE

  Threshold Call: 1
  Number Above: 100
  Proportion Above: 1

Estimates
  shape
  0.05323

Standard Error Type: observed

Standard Errors
  shape
  0.08002

Asymptotic Variance Covariance
  shape
  shape 0.006402

Optimization Information
  Convergence: successful
  Function Evaluations: 17
  Gradient Evaluations: 6

If now, we want to fit a GPD with a varying threshold, just do:

> x <- rgpd(500, 1:2, 0.3, 0.01)
> fitgpd(x, 1:2, est = "mle")
Estimator: MLE
Deviance: -118.8252
AIC: -114.8252

Varying Threshold: TRUE

Threshold Call: 1:2
Number Above: 500
Proportion Above: 1

Estimates
scale  shape
0.3233  0.0103

Standard Error Type: observed

Standard Errors
scale  shape
0.02049  0.04491

Asymptotic Variance Covariance
scale  shape
scale  0.0004199 -0.0006454
shape -0.0006454  0.0020169

Optimization Information
Convergence: successful
Function Evaluations: 46
Gradient Evaluations: 12

Note that the varying threshold is repeated cyclically until it matches the length of object x.

3.3.2 The bivariate case

The generic function to fit bivariate POT is \texttt{fitbvgpd}. There is currently 6 models for the bivariate GPD - see Annex \[\footnote{A}. All of these models are fitted using maximum likelihood estimator. Moreover, the approach uses censored likelihood - see \textit{[Smith et al] 1997}.

```r
> x <- rgpd(500, 0, 1, 0.25)
> y <- rgpd(500, 2, 0.5, -0.25)
> Mlog <- fitbvgpd(cbind(x,y), c(0,2), model = "log")
> Mlog
```

Call: fitbvgpd(data = cbind(x, y), threshold = c(0, 2), model = "log")
Estimator: MLE
Dependence Model and Strength:
  Model : Logistic
  lim_u Pr[ X_1 > u | X_2 > u] = 0.028
Deviance: 1384.825
AIC: 1394.825

Marginal Threshold: 0 2
Marginal Number Above: 500 500
Marginal Proportion Above: 1 1
Joint Number Above: 500
Joint Proportion Above: 1
Number of events such as \{Y1 > u1\} \cup \{Y2 > u2\}: 500
Estimates
scale1 shape1 scale2 shape2 alpha
0.9594 0.3237 0.5468 -0.2973 0.9796

Standard Errors
scale1 shape1 scale2 shape2 alpha
0.06958 0.05833 0.03396 0.04472 0.01947

Asymptotic Variance Covariance
scale1 shape1 scale2 shape2 alpha
scale1 4.841e-03 -2.488e-03 5.653e-05 -6.794e-05 -2.785e-05
shape1 -2.488e-03 3.403e-03 -1.336e-05 8.440e-05 7.683e-07
scale2 5.653e-05 -1.336e-05 1.153e-03 -1.333e-03 -5.972e-06
shape2 -6.794e-05 8.440e-05 -1.333e-03 2.000e-03 7.151e-06
alpha -2.785e-05 7.683e-07 -5.972e-06 7.151e-06 3.791e-04

Optimization Information
Convergence: successful
Function Evaluations: 52
Gradient Evaluations: 11

In the summary, we can see \( \lim_u \Pr[X_1 > u \mid X_2 > u] = 0.02 \). This is the \( \chi \) statistics of Coles et al. (1999). For the parametric model, we have:

\[
\chi = 2 - V(1,1) = 2 \left(1 - A(0.5)\right)
\]

For independent variables, \( \chi = 0 \) while for perfect dependence, \( \chi = 1 \). In our application, the value 0.02 indicates that the variables are independent – which is obvious. In this perspective, it is possible to fix some parameters. For our purpose of independence, we can run -which is equivalent to fit x and y separately of course:

\[
\text{> fitbvgpd(cbind(x,y), c(0,2), model = "log", alpha = 1)}
\]

Call: fitbvgpd(data = cbind(x, y), threshold = c(0, 2), model = "log", alpha = 1)
Estimator: MLE
Dependence Model and Strength:
Model : Logistic
\[
\lim_u \Pr[X_1 > u \mid X_2 > u] = 0
\]
Deviance: 1386.173
AIC: 1394.173
Marginal Threshold: 0 2
Marginal Number Above: 500 500
Marginal Proportion Above: 1 1
Joint Number Above: 500
Joint Proportion Above: 1
Number of events such as \{Y1 > u1\} U \{Y2 > u2\}: 500

Estimates
scale1 shape1 scale2 shape2
0.9567 0.3291 0.5461 -0.2938

Standard Errors
scale1 shape1 scale2 shape2
0.06950 0.05911 0.03410 0.04550

Asymptotic Variance Covariance
Figure 5: The Pickands’ dependence function

```plaintext
scale1  shape1  scale2  shape2
scale1  4.831e-03  -2.506e-03  2.055e-12  -2.365e-12
shape1  -2.506e-03  3.494e-03  -9.091e-13  1.013e-12
scale2  2.055e-12  -9.091e-13  1.163e-03  -1.357e-03
shape2  -2.365e-12  1.013e-12  -1.357e-03  2.071e-03
```

Optimization Information
Convergence: successful
Function Evaluations: 42
Gradient Evaluations: 8

Note that as all bivariate extreme value distributions are asymptotically dependent, the \( \chi \) statistic of Coles et al. (1999) is always equal to 1.

Another way to detect the strength of dependence is to plot the Pickands’ dependence function – see Figure 5. This is simply done with the `pickdep` function.

```R
> pickdep(Mlog)
```

The horizontal line corresponds to independence while the other ones correspond to perfect dependence. Please note that by construction, the `mixed` and `asymmetric mixed` models can not model perfect dependence variables.
3.3.3 Markov Chains for Exceedances

The classical way to perform an analysis of peaks over a threshold is to fit the GPD to cluster maxima. However, there is a waste of data as only the cluster maxima is considered. On the contrary, if we fit the GPD to all exceedances, standard errors are underestimated as we consider independence for dependent observations. Here is where Markov Chains can help us. The main idea is to model the dependence structure using a Markov Chains while the joint distribution is obviously a multivariate extreme value distribution. This idea was first introduced by Smith et al. (1997).

In the remainder of this section, we will only focus with first order Markov Chains. Thus, the likelihood for all exceedances is:

\[
L(y_1, \ldots, y_n; \theta, \psi) = \prod_{i=2}^{n} f(y_i - 1, y_i; \theta, \psi) \prod_{i=2}^{n-1} f(y_i; \theta) \tag{3.12}
\]

where \( f(y_{i-1}, y_i; \theta, \psi) \) is the joint density, \( f(y_i; \theta) \) is the marginal density, \( \theta \) is the marginal GPD parameters and \( \psi \) is the dependence parameter. The marginals are modelled using a GPD, while the joint distribution is a bivariate extreme value distribution.

For our application, we use the `simmc` function which simulate a first order Markov chain with extreme value dependence structure.

```r
> mc <- simmc(1000, alpha = 0.5, model = "log")
> mc <- qgpd(mc, 2, 1, 0.15)
> fitmcgpd(mc, 2, "log")
```

Call: fitmcgpd(data = mc, threshold = 2, model = "log")
Estimator: MLE
Dependence Model and Strength:
  Model : Logistic
    \( \lim_{u} \Pr[X_1 > u \mid X_2 > u] = 0.572 \)
Deviance: 1388.349
  AIC: 1394.349

Threshold Call:
Number Above: 998
Proportion Above: 1

Estimates
  scale  shape  alpha
0.9204  0.1252  0.5139

Standard Errors
  scale  shape  alpha
0.08689  0.03973  0.02221

Asymptotic Variance Covariance
  scale  shape  alpha
scale  0.0075498  -0.0014100  -0.0010378
shape  -0.0014100  0.0015781  -0.0002856
alpha  -0.0010378  -0.0002856  0.0004932

Optimization Information
  Convergence: successful
  Function Evaluations: 52
  Gradient Evaluations: 13
3.4 Confidence Intervals

Once a statistical model is fitted, it is usual to give confidence intervals. Currently, only \texttt{mle}, \texttt{pwmu}, \texttt{pwmb}, \texttt{moments} estimators can compute confidence intervals. Moreover, for method \texttt{mle}, “standard” and “profile” confidence intervals are available.

If we want confidence intervals for the scale parameters:

```r
x <- rgpd(200, 1, 2, 0.25)
mle <- fitgpd(x, 1, est = "mle")
mom <- fitgpd(x, 1, est = "moments")
pwmb <- fitgpd(x, 1, est = "pwmb")
pwmu <- fitgpd(x, 1, est = "pwmu")
gpd.fiscale(mle, conf = 0.9)
```

<table>
<thead>
<tr>
<th>conf.inf.scale</th>
<th>conf.sup.scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.825177</td>
<td>2.693660</td>
</tr>
</tbody>
</table>

```r
gpd.fiscale(mom, conf = 0.9)
```

<table>
<thead>
<tr>
<th>conf.inf.scale</th>
<th>conf.sup.scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.916348</td>
<td>2.990749</td>
</tr>
</tbody>
</table>

```r
gpd.fiscale(pwmu, conf = 0.9)
```

<table>
<thead>
<tr>
<th>conf.inf.scale</th>
<th>conf.sup.scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.814801</td>
<td>2.631560</td>
</tr>
</tbody>
</table>

```r
gpd.fiscale(pwmb, conf = 0.9)
```

<table>
<thead>
<tr>
<th>conf.inf.scale</th>
<th>conf.sup.scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.825151</td>
<td>2.645292</td>
</tr>
</tbody>
</table>

For shape parameter confidence intervals, simply use function \texttt{gpd.fishape} instead of \texttt{gpd.fiscale}. Note that the \texttt{fi} stands for “Fisher Information”.

Thus, if we want profile confidence intervals, we must use functions \texttt{gpd.pfscale} and \texttt{gpd.pfshape}. The \texttt{pf} stands for “profile”. These functions are only available with a model fitted with MLE.

```r
par(mfrow=c(1,2))
gpd.pfscale(mle, range = c(1, 2.9), conf = 0.9)
```

If there is some troubles try to put \texttt{vert.lines = FALSE} or change the range...

<table>
<thead>
<tr>
<th>conf.inf</th>
<th>conf.sup</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.854040</td>
<td>2.736869</td>
</tr>
</tbody>
</table>

```r
gpd.pfshape(mle, range = c(0, 0.6), conf = 0.85)
```

If there is some troubles try to put \texttt{vert.lines = FALSE} or change the range...

<table>
<thead>
<tr>
<th>conf.inf</th>
<th>conf.sup</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.100000</td>
<td>0.372727</td>
</tr>
</tbody>
</table>

Confidence interval for quantiles - or return levels - are also available. This is achieved using: (a) the Delta method or (b) profile likelihood.
If there is some troubles try to put vert.lines = FALSE or change
the range...
conf.inf  conf.sup
1.854040  2.736869

If there is some troubles try to put vert.lines = FALSE or change
the range...
conf.inf  conf.sup
0.1000000  0.3727273

Figure 6: The profile log-likelihood confidence intervals
If there is some troubles try to put vert.lines = FALSE or change the range.

```
> gpd.firl(pwmu, prob = 0.95)
conf.inf  conf.sup
 8.450986  12.777347

> gpd.pfrl(mle, prob = 0.95, range = c(5, 16))

conf.inf  conf.sup
 8.944444  13.722222
```

The profile confidence interval functions both returns the confidence interval and plot the profile log-likelihood function. Figure 7 depicts the graphic window returned by function `gpd.pfrl` for the return level associated to non exceedence probability 0.95.
3.5 Model Checking

To check the fitted model, users must call function `plot` which has a method for the `uvpot`, `bypot` and `mcpot` classes. For example, this is a generic function which calls functions: `pp` (probability/probability plot), `qq` (quantile/quantile plot), `dens` (density plot) and `retlev` (return level plot) for the `uvpot` class.

Here is a basic illustration of the function `plot` for the class `uvpot`.

```r
> x <- rgpd(200, 10, 0.5, -0.2)

> fitted <- fitgpd(x, 10, est = "mle")
> par(mfrow=c(2,2))
> plot(fitted, npy = 1)
```

Figure 8 displays the graphic windows obtained with the latter execution.

If one is interested in only a probability/probability plot, there is two options. We can call function `pp` or equivalently `plotgpd` with the `which` option. The “which” option select which graph you want to plot. That is:

- `which = 1` for a probability/probability plot;
- `which = 2` for a quantile/quantile plot;
- `which = 3` for a density plot;

![Figure 8: Graphical diagnostic for a fitted POT model (univariate case)](image-url)
which = 4 for a return level plot;

Note that “which” can be a vector like c(1,3) or 1:3.
Thus, the following instruction gives the same graphic.

```r
> plot(fitted, which = 1)
> pp(fitted)
```

If a return level plot is asked (4 ∈ which), a value for npy is needed. “npy” corresponds to the mean number of events per year. This is required to define the “return period”. If missing, the default value (i.e. 1) will be chosen.

### 3.6 Declustering Techniques

In opposition to block maxima, a peak over threshold can be problematic when dealing with time series. Indeed, as often time series are strongly auto-correlated, select naively events above a threshold may lead to dependent events.

The function `clust` tries to identify peaks over a threshold while meeting independence criteria. For this purpose, this function needs at least two arguments: the threshold u and a time condition for independence `tim.cond`. Clusters are identify as follow:

1. The first exceedance initiates the first cluster;
2. The first observation under the threshold u “ends” the cluster unless `tim.cond` does not hold;
3. The next exceedance which hold `tim.cond` initiates a new cluster;
4. The process is iterated as needed.

Here is an application on flood discharges for river Ardière at Beaujeu. A preliminary study shows that two flood events can be considered independent if they do not lie within a 8 days window. Note that unit to define `tim.cond` must be the same than the data analyzed.

```r
> data(ardieres)
> events <- clust(ardieres, u = 2, tim.cond = 8 / 365)
```

Several options can be passed to the “clust” function. By default, it will return a list with the identified clusters. Usually, we want only cluster maxima, this is achieved by passing option `clust.max = TRUE`. Users can also ask for a graphic representation of clusters by passing option `plot = TRUE` - see Figure 9.

```r
> clustMax <- clust(ardieres, u = 2, tim.cond = 8 / 365, clust.max = TRUE, plot = TRUE, xlim = c(1971.1, 1972.9))
```

### 3.7 Miscellaneous functions

#### 3.7.1 Return periods: `rp2prob` and `prob2rp`

The functions `rp2prob` and `prob2rp` are useful to convert return periods to non exceedence probabilities and vice versa. It needs either a return period either a non exceedence probability. Moreover, the mean number of events per year “npy” must be specified.

```r
> rp2prob(50, 1.8)

npy retper  prob
1 1.8    50 0.9888889

> prob2rp(0.6, 2.2)

npy retper  prob
1 2.2    1.138364 0.6
```
Figure 9: The identified clusters. Data Ardières, u = 2, tim.cond = 8
Figure 10: Instantaneous flood discharges and averaged discharged over duration 3 days. Data ardieres

3.7.2 Unbiased Sample L-Moments: `samlmu`

The function `samlmu` computes the unbiased sample L-Moments.

```r
> x <- runif(50)
> samlmu(x, nmom = 5)
```

<table>
<thead>
<tr>
<th>l_1</th>
<th>l_2</th>
<th>t_3</th>
<th>t_4</th>
<th>t_5</th>
</tr>
</thead>
</table>
| 0.46606092 | 0.16884217 | 0.02524530 | 0.1716963 | 0.02876797 

3.7.3 Mobile average window on time series: `ts2tsd`

The function `ts2tsd` computes an “average” time series tsd from the initial time series ts. This is achieved by using a mobile average window of length d on the initial time series.

```r
> data(ardieres)
> tsd <- ts2tsd(ardieres, 3 / 365)
> plot(ardieres, type = "l", col = "blue")
> lines(tsd, col = "green")
```

The latter execution is depicted in Figure 10.
4 A Concrete Statistical Analysis of Peaks Over a Threshold

In this section, we provide a full and detailed analysis of peaks over a threshold for the river Ardières at Beaujeu. Figure 10 depicts instantaneous flood discharges - blue line.

As this is a time series, we must selects independent events above a threshold. First, we fix a relatively low threshold to “extract” more events. Thus, some of them are not extreme but regular events. This is necessary to select a reasonable threshold for the asymptotic approximation by a GPD - see section 2.

> summary(ardieres)

<table>
<thead>
<tr>
<th></th>
<th>obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min.</td>
<td>0.022</td>
</tr>
<tr>
<td>1st Qu.</td>
<td>0.236</td>
</tr>
<tr>
<td>Median</td>
<td>0.542</td>
</tr>
<tr>
<td>Mean</td>
<td>1.024</td>
</tr>
<tr>
<td>3rd Qu.</td>
<td>1.230</td>
</tr>
<tr>
<td>Max.</td>
<td>44.200</td>
</tr>
<tr>
<td>NA's</td>
<td>1</td>
</tr>
</tbody>
</table>

> events0 <- clust(ardieres, u = 1.5, tim.cond = 8/365, clust.max = TRUE)
> par(mfrow=c(2,2))
> mrlplot(events0[,"obs"])
> abline( v = 6, col = "green")
> diplot(events0)
> abline( v = 6, col = "green")
> tcplot(events0[,"obs"], which = 1)
> abline( v = 6, col = "green")
> tcplot(events0[,"obs"], which = 2)
> abline( v = 6, col = "green")

From Figure 11 a threshold value of $6m^3/s$ should be reasonable. The Mean residual life plot - top left panel- indicates that a threshold around $10m^3/s$ should be adequate. However, the selected threshold must be low enough to have enough events above it to reduce variance while not too low as it increase the bias.

Thus, we can now “re-extract” events above the threshold $6m^3/s$, obtaining object events1. This is necessary as sometimes events1 is not equal to observations of events0 greater than $6m^3/s$. We can now define the mean number of events per year “npy”. Note that an estimation of the extremal index is available.

> events1 <- clust(ardieres, u = 6, tim.cond = 8/365, clust.max = TRUE)
> npy <- length(events1[,"obs"]) / (diff(range(ardieres[,"time"], na.rm = TRUE)) - diff(ardieres[c(20945,20947),"time"]))
> ##Because there is a gap !!!!
> print(npy)

[1] 1.707897

> attributes(events1)$exi

[1] 0.1247265

Let’s fit the GPD.

\[3\]As the asymptotic approximation by a GPD is not accurate anymore.
Figure 11: Threshold selection for river Ardières at Beaujeu.
The result of function `fitgpd` gives the name of the estimator, if a varying threshold was used, the threshold value, the number and the proportion of observations above the threshold, parameter estimates, standard error estimates and type, the asymptotic variance-covariance matrix and convergence diagnostic.

Figure 12 shows graphic diagnostics for the fitted model. It can be seen that the fitted model “mle” seems to be appropriate. Suppose we want to know the return level associated to the 100-year return period.

```r
> mle <- fitgpd(events1[,"obs"], thresh = 6, est = "mle")
> par(mfrow=c(2,2))
> plot(mle, npy = npy)
```

```r
> # First convert return period in prob
> rp2prob(retper = 100, npy = npy)
```

```
 npy  retper   prob
1  1.707897  100  0.9941448
```

```r
> prob <- rp2prob(retper = 100, npy = npy)[,"prob"]
> qgpd(prob, loc = 6, scale = mle$param["scale"], shape = mle$param["shape"])
```

```
scale
36.44331
```

To take into account uncertainties, Figure 13 depicts the profile confidence interval for the quantile associated to the 100-year return period.

Figure 12: Graphic diagnostics for river Ardières at Beaujeu
If there is some troubles try to put vert.lines = FALSE or change the range...

conf.inf  conf.sup
25.56533 90.76633

Sometimes it is necessary to know the estimated return period of a specified events. Lets do it with the larger events in “events1”.

```r
> maxEvent <- max(events1[, "obs"])
> maxEvent

[1] 44.2
```

```r
> prob <- pgpd(maxEvent, loc = 6, scale = mle$param["scale"], shape =
+ mle$param["shape"])
> prob
```
shape
0.9974115

> prob2rp(prob, npy = npy)

     npy  retper    prob
1 1.707897 226.1982 0.9974115

Thus, the largest events that occurs in June 2000 has approximately a return period of 240 years. Maybe it is a good idea to fit the GPD with the other estimators available in the POT package.
A Dependence Models for Bivariate Extreme Value Distributions

A.1 The Logistic model

The logistic model is defined by:

\[ V(x, y) = \left( x^{-1/\alpha} + y^{-1/\alpha} \right)^\alpha, \quad 0 < \alpha \leq 1 \quad (A.1) \]

Independence is obtained when \( \alpha = 1 \) while total dependence for \( \alpha \to 0 \).

The Pickands’ dependence function for the logistic model is:

\[ A: [0, 1] \longrightarrow [0, 1] \]

\[ w \longrightarrow \left[ (1 - w)^{1/\alpha} + w^{1/\alpha} \right]^\alpha \]

A.2 The Asymetric Logistic model

The asymetric logistic model is defined by:

\[ V(x, y) = 1 - \theta_1 x + 1 - \theta_2 y + \left[ \left( \frac{x}{\theta_1} \right)^{-\frac{1}{\alpha}} + \left( \frac{y}{\theta_2} \right)^{-\frac{1}{\alpha}} \right]^\alpha, \]

with \( 0 < \alpha \leq 1 \), \( 0 \leq \theta_1, \theta_2 \leq 1 \).

Independence is obtained when either \( \alpha = 1 \), \( \theta_1 = 0 \) or \( \theta_2 = 0 \). Differents limits occur when \( \theta_1 \) and \( \theta_2 \) are fixed and \( \alpha = 1 \to 0 \).

The Pickands’ dependence function for the asymetric logistic model is:

\[ A(w) = (1 - \theta_1)(1 - w) + (1 - \theta_2)w + \left[ (1 - w)^{1/\alpha} \theta_1^{1/\alpha} + w^{1/\alpha} \theta_2^{1/\alpha} \right]^\alpha \]

A.3 The Negative Logistic model

The negative logistic model is defined by:

\[ V(x, y) = 1 - \frac{1}{x} + \frac{1}{y} - (x^\alpha + y^\alpha)^{-\frac{1}{\alpha}}, \quad \alpha > 0 \quad (A.2) \]

Independence is obtained when \( \alpha \to 0 \) while total dependence when \( \alpha \to +\infty \).

The Pickands’ dependence function for the negative logistic model is:

\[ A(w) = 1 - \left[ (1 - w)^{-\alpha} + w^{-\alpha} \right]^{-\frac{1}{\alpha}} \]

A.4 The Asymetric Negative Logistic model

The asymetric negative logistic model is defined by:

\[ V(x, y) = 1 - \frac{1}{x} + \frac{1}{y} - \left[ \left( \frac{x}{\theta_1} \right)^{\alpha} + \left( \frac{y}{\theta_2} \right)^{\alpha} \right]^{-\frac{1}{\alpha}}, \quad \alpha > 0, \quad 0 < \theta_1, \theta_2 \leq 1 \]

Independence is obtained when either \( \alpha \to 0 \), \( \theta_1 \to 0 \) or \( \theta_2 \to 0 \). Differents limits occur when \( \theta_1 \) and \( \theta_2 \) are fixed and \( \alpha \to +\infty \).

The Pickands’ dependence function for the asymetric negative logistic model is:

\[ A(w) = 1 - \left[ \left( \frac{1 - w}{\theta_1} \right)^{-\alpha} + \left( \frac{w}{\theta_2} \right)^{-\alpha} \right]^{-\frac{1}{\alpha}} \]
A.5 The Mixed model

The mixed model is defined by:

\[ V(x, y) = \frac{1}{x} + \frac{1}{y} - \frac{\alpha}{x + y}, \quad 0 \leq \alpha \leq 1 \]

Independence is obtained when \( \alpha = 0 \) while total dependence could never be reached.

The Pickands’ dependence function for the mixed model is:

\[ A(w) = 1 - w (1 - w) \alpha \]

A.6 The Asymmetric Mixed model

The asymmetric mixed model is defined by:

\[ V(x, y) = \frac{1}{x} + \frac{1}{y} - \frac{(\alpha + \theta) x + (\alpha + 2\theta) y}{(x + y)^2}, \quad \alpha \geq 0, \quad \alpha + 2\theta \leq 1, \quad \alpha + 3\theta \geq 0 \]

Independence is obtained when \( \alpha = \theta = 0 \) while total dependence could never be reached.

The Pickands’ dependence function for the asymmetric mixed model is:

\[ A(w) = \theta w^3 + \alpha w^2 - (\alpha + \theta) w + 1 \]

References


