Diagrams and Procedures for Partition of Variation

Pierre Legendre

processed with vegan 2.3-5 in R version 3.2.4 Patched (2016-03-28 r70390) on April 4, 2016

Diagrams describing the partitions of variation of a response data table by two (Fig. 1), three (Fig. 2) and four tables (Fig. 3) of explanatory variables. The fraction names [a] to [p] in the output of `varpart` function follow the notation in these Venn diagrams, and the diagrams were produced using the `showvarparts` function.

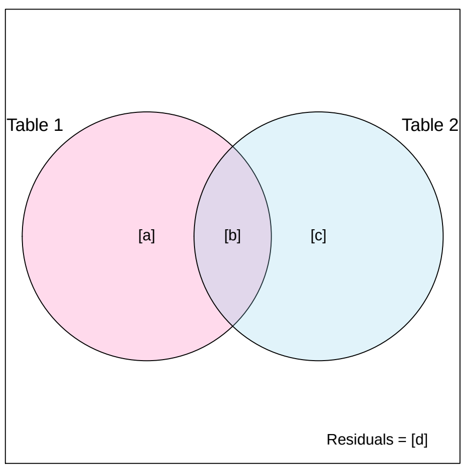

Figure 1: 3 regression/ canonical analyses and 3 subtraction equations are needed to estimate the 4 (= 2^2) fractions.
[a] and [c] can be tested for significance (3 canonical analyses per permutation). Fraction [b] cannot be tested singly.

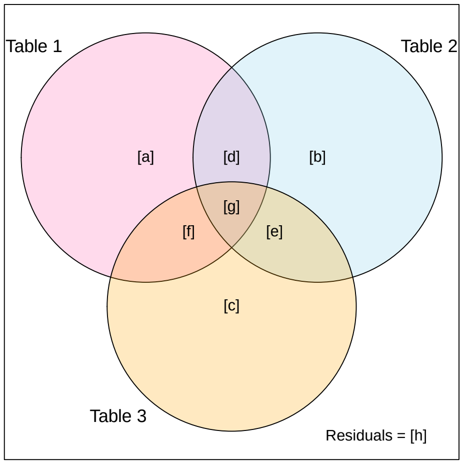

Figure 2: 7 regression/canonical analyses and 10 subtraction equations are needed to estimate the 8 ($= 2^3$) fractions.
[a] to [c] and subsets containing [a] to [c] can be tested for significance (4 canonical analyses per permutation to test [a] to [c]). Fractions [d] to [g] cannot be tested singly.

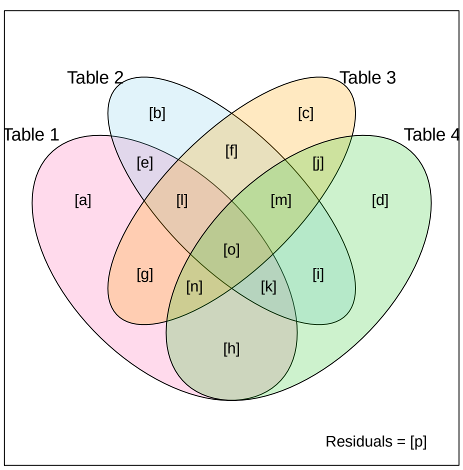

Figure 3: 15 regression/canonical analyses and 27 subtraction equations are needed to estimate the 16 ($= 2^4$) fractions.
[a] to [d] and subsets containing [a] to [d] can be tested for significance (5 canonical analyses per permutation to test [a] to [d]). Fractions [e] to [o] cannot be tested singly.

Variation partitioning for two explanatory data tables --
Table 1 with m1 explanatory variables, Table 2 with m2 explanatory variables
Number of fractions: 4, called [a] ... [d]
√ indicates the 3 regression or canonical analyses that have to be computed.
Partial canonical analyses are only computed if tests of significance or biplots are needed.

```
Compute   Fitted  Residuals   Derived fractions                 Degrees of freedom, numerator of F

√ Y.1     [a+b]     [c+d] (1)                                   df(a+b) = m1
√ Y.2     [b+c]     [a+d] (2)                                   df(b+c) = m2
√ Y.1,2   [a+b+c]    [d]  (3)                                   df(a+b+c) = m3 ≤ m1+m2 (there may be collinearity)
# Y.1|2   [a]        [d]                                        df(a) = m3-m2
# Y.2|1   [c]        [d]                                        df(c) = m3-m1

Partial analyses          (4) [a] = [a+b+c] - [b+c]             df(a) = m3-m2*
controlling for 1 table X (5) [c] = [a+b+c] - [a+b]             df(c) = m3-m1*
                          (6) [b] = [a+b] + [b+c] - [a+b+c]     df(b) = m1+m2-(m1+m2) = 0
                          (7) [d] = residuals = 1 - [a+b+c]     df2(d) = n-1-m3 for denominator of F
```

* Calculation of d.f. for difference between nested models: see Sokal & Rohlf (1981, 1995) equation 16.14.

Tests of significance --

```
F(a+b) = ([a+b]/m1)/([c+d]/(n-1-m1))
F(b+c) = ([b+c]/m2)/([a+d]/(n-1-m2))
F(a+b+c) = ([a+b+c]/m3)/([d]/(n-1-m3))

F(a) = ([a]/(m3-m2))/([d]/(n-1-m3))
F(c) = ([c]/(m3-m1))/([d]/(n-1-m3))
```

The only testable fractions are those that can be obtained directly by regression or canonical analysis.
The non-testable fraction is [b]. That fraction cannot be obtained directly by regression or canonical analysis.

Variation partitioning for three explanatory data tables --
Table 1 with m1 explanatory variables, Table 2 with m2 explanatory variables, Table3 with m3 explanatory variables
Number of fractions: 8, called [a] ... [h]
√ indicates the 7 regression or canonical analyses that have to be computed.
Partial canonical analyses are only computed if tests of significance or biplots are needed.

```
Compute   Fitted       Residuals  Derived fractions                        Degrees of freedom, numerator of F

Direct canonical analysis
√ Y.1     [a+d+f+g]     [b+c+e+h] (1)                                       df(a+d+f+g) = m1
√ Y.2     [b+d+e+g]     [a+c+f+h] (2)                                       df(b+d+e+g) = m2
√ Y.3     [c+e+f+g]     [a+b+d+h] (3)                                       df(c+e+f+g) = m3
√ Y.1,2   [a+b+d+e+f+g]     [c+h] (4)                                       df(a+b+d+e+f+g) = m4 ≤ m1+m2 (collinearity?)
√ Y.1,3   [a+c+d+e+f+g]     [b+h] (5)                                       df(a+c+d+e+f+g) = m5 ≤ m1+m3 (collinearity?)
√ Y.2,3   [b+c+d+e+f+g]     [a+h] (6)                                       df(b+c+d+e+f+g) = m6 ≤ m2+m3 (collinearity?)
√ Y.1,2,3 [a+b+c+d+e+f+g]     [h] (7)                                       df(a+b+c+d+e+f+g) = m7 ≤ m1+m2+m3 (collinearity?)
# Y.1|2   [a+f]             [c+h]                                           df(a+f) = m4-m2
# Y.1|3   [a+d]             [b+h]                                           df(a+d) = m5-m3
# Y.2|1   [b+e]             [c+h]                                           df(b+e) = m4-m1
# Y.2|3   [b+d]             [a+h]                                           df(b+d) = m6-m3
# Y.3|1   [c+e]             [b+h]                                           df(c+e) = m5-m1
# Y.3|2   [c+f]             [a+h]                                           df(c+f) = m6-m2
# Y.1|2,3 [a]                 [h]                                           df(a) = m7-m6
# Y.2|1,3 [b]                 [h]                                           df(b) = m7-m5
# Y.3|1,2 [c]                 [h]                                           df(c) = m7-m4

Partial analyses               (8) [a] = [a+b+c+d+e+f+g] - [b+c+d+e+f+g]    df(a) = m7-m6
controlling for two tables X   (9) [b] = [a+b+c+d+e+f+g] - [a+c+d+e+f+g]    df(b) = m7-m5
                              (10) [c] = [a+b+c+d+e+f+g] - [a+b+d+e+f+g]    df(c) = m7-m4

controlling for one table X   (11) [a+d] = [a+c+d+e+f+g] - [c+e+f+g]        df(a+d) = m5-m3
                              (12) [a+f] = [a+b+d+e+f+g] - [b+d+e+g]        df(a+f) = m4-m2
                              (13) [b+d] = [b+c+d+e+f+g] - [c+e+f+g]        df(b+d) = m6-m3
                              (14) [b+e] = [a+b+d+e+f+g] - [a+d+f+g]        df(b+e) = m4-m1
                              (15) [c+e] = [a+c+d+e+f+g] - [a+d+f+g]        df(c+e) = m5-m1
                              (16) [c+f] = [b+c+d+e+f+g] - [b+d+e+g]        df(c+f) = m6-m2

Fractions estimated           (17) [d] = [a+d] - [a]                        df(d) = m1-m1 = 0
by subtraction                (18) [e] = [b+e] - [b]                        df(e) = m2-m2 = 0
(cannot be tested)            (19) [f] = [c+f] - [c]                        df(f) = m3-m3 = 0
                              (20) [g] = [a+b+c+d+e+f+g]-[a+d]-[b+e]-[c+f]  df(g) = (m1+m2+m3)-m1-m2-m3 = 0
                                or [g] = [a+d+f+g] - [a] - [d] - [f]        df(g) = m1-m1-0-0 = 0
                              (21) [h] = residuals = 1 - [a+b+c+d+e+f+g]    df2(h) = n-1-m7 for denominator of F
```

Tests of significance --

```
F(a+d+f+g) = ([a+d+f+g]/m1)/([b+c+e+h]/(n-1-m1))
F(b+d+e+g) = ([b+d+e+g]/m2)/([a+c+f+h]/(n-1-m2))
F(c+e+f+g) = ([c+e+f+g]/m3)/([a+b+d+h]/(n-1-m3))
F(a+b+d+e+f+g) = ([a+b+d+e+f+g]/m4)/([c+h]/(n-1-m4))
F(a+c+d+e+f+g) = ([a+c+d+e+f+g]/m5)/([b+h]/(n-1-m5))
F(b+c+d+e+f+g) = ([b+c+d+e+f+g]/m6)/([a+h]/(n-1-m6))
F(a+b+c+d+e+f+g) = ([a+b+c+d+e+f+g]/m7)/([h]/(n-1-m7))

F(a) = ([a]/(m7-m6))/([h]/(n-1-m7))
F(b) = ([b]/(m7-m5))/([h]/(n-1-m7))
F(c) = ([c]/(m7-m4))/([h]/(n-1-m7))
F(a+d) = ([a+d]/(m5-m3))/([b+h]/(n-1-m5))
F(a+f) = ([a+f]/(m4-m2))/([c+h]/(n-1-m4))
F(b+d) = ([b+d]/(m6-m3))/([a+h]/(n-1-m6))
F(b+e) = ([b+e]/(m4-m1))/([c+h]/(n-1-m4))
F(c+e) = ([c+e]/(m5-m1))/([b+h]/(n-1-m5))
F(c+f) = ([c+f]/(m6-m2))/([a+h]/(n-1-m6))
```

The only testable fractions are those that can be obtained directly by regression or canonical analysis.

```
Variation partitioning for four explanatory data tables --
Table 1 with m1 variables, Table 2 with m2 variables, Table3 with m3 variables, Table4 with m4 variables
Number of fractions: 16, called [a] ... [p].
√ indicates the 15 regression or canonical analyses that have to be computed.

Compute     Fitted                        Residuals         Derived fractions                                         Degrees of freedom

Direct canonical analysis
√ Y.1       [a+e+g+h+k+l+n+o]             [b+c+d+f+i+j+m+p] (1)                                                       df(a+e+g+h+k+l+n+o) = m1
√ Y.2       [b+e+f+i+k+l+m+o]             [a+c+d+g+h+j+n+p] (2)                                                       df(b+e+f+i+k+l+m+o) = m2
√ Y.3       [c+f+g+j+l+m+n+o]             [a+b+d+e+h+i+k+p] (3)                                                       df(c+f+g+j+l+m+n+o) = m3
√ Y.4       [d+h+i+j+k+m+n+o]             [a+b+c+e+f+g+l+p] (4)                                                       df(d+h+i+j+k+m+n+o) = m4
√ Y.1,2     [a+b+e+f+g+h+i+k+l+m+n+o]             [c+d+j+p] (5)                                                       df(a+b+e+f+g+h+i+k+l+m+n+o) = m5  ≤ m1+m2
√ Y.1,3     [a+c+e+f+g+h+j+k+l+m+n+o]             [b+d+i+p] (6)                                                       df(a+c+e+f+g+h+j+k+l+m+n+o) = m6  ≤ m1+m3
√ Y.1,4     [a+d+e+g+h+i+j+k+l+m+n+o]             [b+c+f+p] (7)                                                       df(a+d+e+g+h+i+j+k+l+m+n+o) = m7  ≤ m1+m4
√ Y.2,3     [b+c+e+f+g+i+j+k+l+m+n+o]             [a+d+h+p] (8)                                                       df(b+c+e+f+g+i+j+k+l+m+n+o) = m8  ≤ m2+m3
√ Y.2,4     [b+d+e+f+h+i+j+k+l+m+n+o]             [a+c+g+p] (9)                                                       df(b+d+e+f+h+i+j+k+l+m+n+o) = m9  ≤ m2+m4
√ Y.3,4     [c+d+f+g+h+i+j+k+l+m+n+o]             [a+b+e+p] (10)                                                      df(c+d+f+g+h+i+j+k+l+m+n+o) = m10 ≤ m3+m4
√ Y.1,2,3   [a+b+c+e+f+g+h+i+j+k+l+m+n+o]             [d+p] (11)                                                      df(a+b+c+e+f+g+h+i+j+k+l+m+n+o) = m11 ≤ m1+m2+m3
√ Y.1,2,4   [a+b+d+e+f+g+h+i+j+k+l+m+n+o]             [c+p] (12)                                                      df(a+b+d+e+f+g+h+i+j+k+l+m+n+o) = m12 ≤ m1+m2+m4
√ Y.1,3,4   [a+c+d+e+f+g+h+i+j+k+l+m+n+o]             [b+p] (13)                                                      df(a+c+d+e+f+g+h+i+j+k+l+m+n+o) = m13 ≤ m1+m3+m4
√ Y.2,3,4   [b+c+d+e+f+g+h+i+j+k+l+m+n+o]             [a+p] (14)                                                      df(b+c+d+e+f+g+h+i+j+k+l+m+n+o) = m14 ≤ m2+m3+m4
√ Y.1,2,3,4 [a+b+c+d+e+f+g+h+i+j+k+l+m+n+o]             [p] (15)                                                      df(a+b+c+d+e+f+g+h+i+j+k+l+m+n+o) = m15 ≤ m1+m2+m3+m4

Partial analyses
controlling for one table X                          (16) [a+g+h+n] = [a+b+e+f+g+h+i+k+l+m+n+o] - [b+e+f+i+k+l+m+o]       df(a+g+h+n) = m5 - m2
                                                     (17) [a+e+h+k] = [a+c+e+f+g+h+j+k+l+m+n+o] - [c+f+g+j+l+m+n+o]       df(a+e+h+k) = m6 - m3
                                                     (18) [a+e+g+l] = [a+d+e+g+h+i+j+k+l+m+n+o] - [d+h+i+j+k+m+n+o]       df(a+e+g+l) = m7 - m4

                                                     (19) [b+f+i+m] = [a+b+e+f+g+h+i+k+l+m+n+o] - [a+e+g+h+k+l+n+o]       df(b+f+i+m) = m5 - m1
                                                     (20) [b+e+i+k] = [b+c+e+f+g+i+j+k+l+m+n+o] - [c+f+g+j+l+m+n+o]       df(b+e+i+k) = m8 - m3
                                                     (21) [b+e+f+l] = [b+d+e+f+h+i+j+k+l+m+n+o] - [d+h+i+j+k+m+n+o]       df(b+e+f+l) = m9 - m4

                                                     (22) [c+f+j+m] = [a+c+e+f+g+h+j+k+l+m+n+o] - [a+e+g+h+k+l+n+o]       df(a) = m6 - m1
                                                     (23) [c+g+j+n] = [b+c+e+f+g+i+j+k+l+m+n+o] - [b+e+f+i+k+l+m+o]       df(a) = m8 - m2
                                                     (24) [c+f+g+l] = [c+d+f+g+h+i+j+k+l+m+n+o] - [d+h+i+j+k+m+n+o]       df(a) = m10 - m4

                                                     (25) [d+i+j+m] = [a+d+e+g+h+i+j+k+l+m+n+o] - [a+e+g+h+k+l+n+o]       df(a) = m7 - m1
                                                     (26) [d+h+j+n] = [b+d+e+f+h+i+j+k+l+m+n+o] - [b+e+f+i+k+l+m+o]       df(a) = m9 - m2
                                                     (27) [d+h+i+k] = [c+d+f+g+h+i+j+k+l+m+n+o] - [c+f+g+j+l+m+n+o]       df(a) = m10 - m3

controlling for two tables X                         (28) [a+e] = [a+c+d+e+f+g+h+i+j+k+l+m+n+o]-[c+d+f+g+h+i+j+k+l+m+n+o]     df(a+e) = m13 - m10
                                                     (29) [a+g] = [a+b+d+e+f+g+h+i+j+k+l+m+n+o]-[b+d+e+f+h+i+j+k+l+m+n+o]     df(a+g) = m12 - m9
                                                     (30) [a+h] = [a+b+c+e+f+g+h+i+j+k+l+m+n+o]-[b+c+e+f+g+i+j+k+l+m+n+o]     df(a+h) = m11 - m8

                                                     (31) [b+e] = [b+c+d+e+f+g+h+i+j+k+l+m+n+o]-[c+d+f+g+h+i+j+k+l+m+n+o]     df(b+e) = m14 - m10
                                                     (32) [b+f] = [a+b+d+e+f+g+h+i+j+k+l+m+n+o]-[a+d+e+g+h+i+j+k+l+m+n+o]     df(b+f) = m12 - m7
                                                     (33) [b+i] = [a+b+c+e+f+g+h+i+j+k+l+m+n+o]-[a+c+e+f+g+h+j+k+l+m+n+o]     df(b+i) = m11 - m6

                                                     (34) [c+f] = [a+c+d+e+f+g+h+i+j+k+l+m+n+o]-[a+d+e+g+h+i+j+k+l+m+n+o]     df(c+f) = m13 - m7
                                                     (35) [c+g] = [b+c+d+e+f+g+h+i+j+k+l+m+n+o]-[b+d+e+f+h+i+j+k+l+m+n+o]     df(c+g) = m14 - m9
                                                     (36) [c+j] = [a+b+c+e+f+g+h+i+j+k+l+m+n+o]-[a+b+e+f+g+h+i+k+l+m+n+o]     df(c+j) = m11 - m5

                                                     (37) [d+h] = [b+c+d+e+f+g+h+i+j+k+l+m+n+o]-[b+c+e+f+g+i+j+k+l+m+n+o]     df(d+h) = m14 - m8
                                                     (38) [d+i] = [a+c+d+e+f+g+h+i+j+k+l+m+n+o]-[a+c+e+f+g+h+j+k+l+m+n+o]     df(d+i) = m13 - m6
                                                     (39) [d+j] = [a+b+d+e+f+g+h+i+j+k+l+m+n+o]-[a+b+e+f+g+h+i+k+l+m+n+o]     df(d+j) = m12 - m5

controlling for three tables X                       (40) [a] = [a+b+c+d+e+f+g+h+i+j+k+l+m+n+o] - [b+c+d+e+f+g+h+i+j+k+l+m+n+o]   df(a) = m15 - m14
                                                     (41) [b] = [a+b+c+d+e+f+g+h+i+j+k+l+m+n+o] - [a+c+d+e+f+g+h+i+j+k+l+m+n+o]   df(b) = m15 - m13
                                                     (42) [c] = [a+b+c+d+e+f+g+h+i+j+k+l+m+n+o] - [a+b+d+e+f+g+h+i+j+k+l+m+n+o]   df(c) = m15 - m12
                                                     (43) [d] = [a+b+c+d+e+f+g+h+i+j+k+l+m+n+o] - [a+b+c+e+f+g+h+i+j+k+l+m+n+o]   df(d) = m15 - m11

Fractions estimated by subtraction                   (44) [e] = [a+e] - [a]                                                   df(e) = m1-m1 = 0
(cannot be tested)                                   (45) [f] = [b+f] - [b]                                                   df(f) = m2-m2 = 0
                                                     (46) [g] = [a+g] - [a]                                                   df(g) = m1-m1 = 0
                                                     (47) [h] = [a+h] - [a]                                                   df(h) = m1-m1 = 0
                                                     (48) [i] = [b+i] - [b]                                                   df(i) = m2-m2 = 0
                                                     (49) [j] = [c+j] - [c]                                                   df(j) = m3-m3 = 0
                                                     (50) [k] = [a+e+h+k] - [a+e] - [h]                                       df(k) = m1-m1-0 = 0
                                                     (51) [l] = [a+e+g+l] - [a+e] - [g]                                       df(l) = m1-m1-0 = 0
                                                     (52) [m] = [b+f+i+m] - [b+f] - [i]                                       df(m) = m2-m2-0 = 0
                                                     (53) [n] = [a+g+h+n] - [a+g] - [h]                                       df(n) = m1-m1-0 = 0
                                                     (54) [o] = [a+e+g+h+k+l+n+o] - [a+e+h+k] - [g] - [l] - [n]               df(o) = m1-m1-0-0-0 = 0
                                                     (55) [p] = residuals = 1 - [a+b+c+d+e+f+g+h+i+j+k+l+m+n+o]               df2(p) = n-1-m15
```

Tests of significance --

```
F(a+e+g+h+k+l+n+o) = ([a+e+g+h+k+l+n+o]/m1)/([b+c+d+f+i+j+m+p]/(n-1-m1))
F(b+e+f+i+k+l+m+o) = ([b+e+f+i+k+l+m+o]/m2)/([a+c+d+g+h+j+n+p]/(n-1-m2))
F(c+f+g+j+l+m+n+o) = ([c+f+g+j+l+m+n+o]/m3)/([a+b+d+e+h+i+k+p]/(n-1-m3))
F(d+h+i+j+k+m+n+o) = ([d+h+i+j+k+m+n+o]/m4)/([a+b+c+e+f+g+l+p]/(n-1-m4))
F(a+b+e+f+g+h+i+k+l+m+n+o) = ([a+b+e+f+g+h+i+k+l+m+n+o]/m5)/([c+d+j+p]/(n-1-m5))
F(a+c+e+f+g+h+j+k+l+m+n+o) = ([a+c+e+f+g+h+j+k+l+m+n+o]/m6)/([b+d+i+p]/(n-1-m6))
F(a+d+e+g+h+i+j+k+l+m+n+o) = ([a+d+e+g+h+i+j+k+l+m+n+o]/m7)/([b+c+f+p]/(n-1-m7))
F(b+c+e+f+g+i+j+k+l+m+n+o) = ([b+c+e+f+g+i+j+k+l+m+n+o]/m8)/([a+d+h+p]/(n-1-m8))
F(b+d+e+f+h+i+j+k+l+m+n+o) = ([b+d+e+f+h+i+j+k+l+m+n+o]/m9)/([a+c+g+p]/(n-1-m9))
F(c+d+f+g+h+i+j+k+l+m+n+o) = ([c+d+f+g+h+i+j+k+l+m+n+o]/m10)/([a+b+e+p]/(n-1-m10))
F(a+b+c+e+f+g+h+i+j+k+l+m+n+o) = ([a+b+c+e+f+g+h+i+j+k+l+m+n+o]/m11)/([d+p]/(n-1-m11))
F(a+b+d+e+f+g+h+i+j+k+l+m+n+o) = ([a+b+d+e+f+g+h+i+j+k+l+m+n+o]/m12)/([c+p]/(n-1-m12))
F(a+c+d+e+f+g+h+i+j+k+l+m+n+o) = ([a+c+d+e+f+g+h+i+j+k+l+m+n+o]/m13)/([b+p]/(n-1-m13))
F(b+c+d+e+f+g+h+i+j+k+l+m+n+o) = ([b+c+d+e+f+g+h+i+j+k+l+m+n+o]/m14)/([a+p]/(n-1-m14))
F(a+b+c+d+e+f+g+h+i+j+k+l+m+n+o) = ([a+b+c+d+e+f+g+h+i+j+k+l+m+n+o]/m15)/([p]/(n-1-m15))

F(a+g+h+n) = ([a+g+h+n]/(m5-m2))/([c+d+j+p]/(n-1-m5))
For the other fractions controlling for one table X, the F-statistics are constructed in the same way

F(a+e) = ([a+e]/(m13-m10))/([b+p]/(n-1-m13))
For the other fractions controlling for two tables X, the F-statistics are constructed in the same way

Fractions controlling for three tables X:
F(a) = ([a]/(m15-m14))/([p]/(n-1-m15))
F(b) = ([b]/(m15-m13))/([p]/(n-1-m15))
F(c) = ([c]/(m15-m12))/([p]/(n-1-m15))
F(d) = ([d]/(m15-m11))/([p]/(n-1-m15))
```

Other fractions combining elementary fractions [a] to [o] can be calculated, but cannot be tested because they cannot be obtained by regression.
